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Series solutions of the non-stationary Heun equation. (English summary)
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This paper is yet another interesting contribution to a novel topic of relations between special functions and integrable models. In this paper, the authors deal with the elliptic generalization of the Calogero-Sutherland model and provide recursive methods for the evaluation of non-stationary solutions in a non-conventional basis.

The work is highly recommended; the first few sections constitute a good review on new solution methods of the Heun—and Lamé—equation, while the last sections present novel contributions regarding computation of the solutions. Of particular importance is Lemma 4.1, whose proof is contained in a previous work [E. Langmann and K. Takemura, J. Math. Phys. **53** (2012), no. 8, 082105; MR3012626]. This lemma ensures the validity of a given kernel as a solution of a non-stationary Schrödinger equation, unconventionally presented as a hyperbolic equation in two space variables (x, y) and parabolic in the space-time pairs (x, τ) and (y, τ). Jacobi theta functions make their appearance in the kernel, as expected, in view of known relations with the Weierstrass elliptic function. The new achievement of this work truly starts with section 5, which is indeed very useful.

With that being said, one has to follow the current literature—cited in the paper—to fully appreciate the value of the results. The formulae cannot be derived by direct and naive calculations and the results rest heavily on integral identities—or source identities—derived previously by one of the authors [op. cit.]. Of particular interest for the reader is the first author's doctoral thesis [*A kernel function approach to exact solutions of Calogero-Moser-Sutherland type models*, KTH Stockholm, 2016], where most of the derivations are explained. Although the introductory text works well as a comprehensive survey, the results must be checked with the aid of more explanatory works as companions.

Regarding some specific hypotheses made throughout this paper, the so-called nomé and its stringent upper bound $|q| < 1$ require some comments. The equation solved by the authors is more of a heat-type equation rather than a Schrödinger-type equation, in all fairness. This is due to the strictly complex time variable lying in the upper semi-plane, and it has an influence on the proofs of convergence for the recursive solutions. Therefore, the extensions of the lemmata presented to a fully oscillatory Schrödinger operator have yet to be studied, perhaps working from the interior of a unit disk, but the final answer seems difficult to unearth. *E. Sadurní*

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