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Ivrii, V. [Ivrii, Victor] (3-TRNT)

Asymptotics of the ground state energy in the relativistic settings. (English summary)

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This paper takes a big step towards sharp asymptotics of relativistic heavy atoms and their statistical properties. The main advance presented here consists in the derivation of a refined upper bound on the ground state energy (and trace formula) containing errors of the type $Z^{3/2}$ for short minimal internuclear distance and $Z^{5/3}$ for long internuclear distance. As announced in the abstract, the corresponding high-order corrections of the Thomas-Fermi approximation are identified as relativistic versions of Scott, Schwinger and Dirac terms. A novelty introduced in the mathematical treatment is the so-called Daubechies inequality, related to lower bounds on square root operators; it is by means of this relation that relativistic kinetic energy operators with mass can be bounded from below, as in (5.2). This improvement is a generalization of the Lieb-Thirring inequality [E. H. Lieb and W. E. Thirring, in *The stability of matter: from atoms to stars*, 135–169, Springer, Berlin, Heidelberg, 1991, doi:10.1007/978-3-662-02725-7˙13], which is applicable to the Laplace operator.

In favour of the present approach, we note that the relativistic proof for the stability of matter is ensured by using naive Hamiltonians such as (1.1) and fermionic states with fixed particle number, such as (1.2). From a more physical perspective, however, one should be aware of the more realistic Dirac equation in relativistic many-body theory, i.e., beyond Chapter 7 of [V. Ivrii, *Microlocal analysis, sharp spectral asymptotics and applications*, 2018]. In particular, the presence of Dirac matrices (and spin) in relativistic wave equations makes the theory more challenging. Despite the existence of a Dirac sea and the apparent system's instability, it is fair to mention that the treatment proposed in [M. Moshinsky and A. G. Nikitin, Rev. Mexicana Fís. **50** (2004), no. 2, suppl., 66–73; MR2117924], in connection with subnuclear structures, should be valid also for multi-electronic atoms. The many-body Foldy-Wouthuysen transformation [L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78** (1950), no. 1, 29–36, doi:10.1103/PhysRev.78.29] is indeed a useful tool that separates particles from holes at the level of the Hamiltonian, with the addition of spin-orbit and Darwin terms that have not been considered in the present paper. For a deeper discussion on the validity of naive models without the full splendor of Quantum Field Theory, see the contributions in [*Relativistic action at a distance: classical and quantum aspects*, Lecture Notes in Physics, 162, Springer, Berlin, 1982; MR0705646].

From the mathematical point of view, the present work is an interesting lesson on functional-analytical techniques. Most of the results can be derived by following previous works sprung from the author's expertise. Regarding the proofs of some inequalities, one is left with an open question on the freedom of rational powers obtained via the p-norm in Hölder's inequality, which is also the case in non-relativistic semiclassics, as in Chapter 25 of [V. Ivrii, op. cit.]. Such a freedom is found, for example, in the proof of the Lieb-Oxford inequality with $p = 4$, and a general value of p may produce slightly different results. In this respect, it is remarkable that the corresponding Hartree-Fock numerical calculations report excellent agreement with asymptotic estimates for the first 55 atoms, as shown in [P. K. Achayra et al., Proc. Nat. Acad. Sci. U.S.A. **77** (1980), no.

12, 6978–6982, doi:10.1073/pnas.77.12.6978]. For the mathematician, it would be also interesting to find shorter proofs for the relativistic case, in parallel to non-relativistic treatments, e.g., in [B. Simon, Astérisque No. 210 (1992), 10, 295–302; MR1221364; W. Hughes, Adv. Math. **79** (1990), no. 2, 213–270; MR1033078]. *E. Sadurní*