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Bispectral dual difference equations for the quantum Toda chain with boundary perturbations. (English summary)

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In the words of Francesco Calogero: "There is only one integrable system", a statement that is nicely illustrated in [V. I. Inozemtsev, Comm. Math. Phys. **121** (1989), no. 4, 629–638; MR0990995]. It is interesting to see the valuable improvements made around the Calogero model over the years, which was introduced in 1971 in the classical domain [F. Calogero, J. Math. Phys. **12** (1971), 419–436; MR0280103; B. Sutherland, Phys. Rev. A **4** (1971), no. 5, 2019–2021, doi:10.1103/PhysRevA.4.2019; Phys. Rev. A **5** (1972), no. 3, 1372–1376, doi:10.1103/PhysRevA.5.1372]. Nowadays it is clear that the quantum Toda system [M. C. Gutzwiller, Ann. Physics **124** (1980), no. 2, 347–381; MR0561316; Ann. Physics **133** (1981), no. 2, 304–331; MR0626693] is but a member within a wider classification of integrable potentials stemming from the Weierstrass elliptic function, as noted by the first author in the work [J. Math. Phys. **36** (1995), no. 3, 1299–1323; MR1317442].

In order to appreciate the contribution of the present paper, one should recognize the usefulness of Darboux transformations in the construction of dual problems that are also integrable [J. J. Duistermaat and F. A. Grünbaum, Comm. Math. Phys. **103** (1986), no. 2, 177–240; MR0826863]. Here, in particular, an infinite-order Darboux transformation containing $\exp(\xi\partial_\lambda)$ may give rise to finite-difference equations in spectral parameters. In this light, the open Toda system with boundary interactions leads to recurrence relations satisfied by hyperoctahedral multivariate Whittaker functions. The authors proceed by recalling the Harish-Chandra series, as well as the construction of the hyperoctahedral Whittaker function with the application of permutations. Difference equations are established in Section 4. Once this is done, the authors study the regularity of the solutions in all variables, with the conclusion that the singularities are removable.

Much work has been done towards a better understanding of many-body quantum integrable systems. The present paper is yet another stepping stone in these explorations.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.