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Double-graded supersymmetric quantum mechanics. (English summary)

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The authors extend supersymmetric quantum mechanics of two supercharges to a higher-dimensional Hilbert space. This is done by the introduction of a $\mathbb{Z}_2^2$ graded structure into the typical shift operators encountered in this formalism; the supercharge, the central charge and the Hamiltonian are now represented by one-dimensional differential operators implanted in 4×4 matrices. These algebraic excursions find a specific realization in the Schrödinger equation of two superpartner Hamiltonians, as discussed by the authors of the present work, together with a reduction to a simple example containing a pair of harmonic oscillators on the real line.

It is amusing to see how old structures appearing in the Dirac equation find their way into new investigations under a different light. The reader may recognize, in these treatments, the introduction of Clifford or Dirac algebras realized as 4×4 matrices—although the authors prefer the analogy with two spins $s = 1/2$ in $\mathfrak{su}(2) + \mathfrak{su}(2)$, as in Section II.D. An extensively studied model in this context is the Dirac oscillator; however, the old nomenclature of this structure is that of a *dynamical supersymmetry*, given the energy-dependent central charge and the fact that a relativistic massless Hamiltonian can be identified with one of the supercharges of the present work [see M. Moshinsky and A. P. Szczepaniak, *J. Phys. A* **22** (1989), no. 17, L817–L819; [MR1012044](#); O. Castaños et al., *Phys. Rev. D* **43** (1991), no. 2, 544–547, [doi:10.1103/PhysRevD.43.544](#); E. Sadurní, J. M. Torres and T. H. Seligman, *J. Phys. A* **43** (2010), no. 28, 285204; [MR2658909](#)].

Other examples of interest may benefit from the matrix representation suggested by the authors. More information on SUSY quantum mechanics applied to the Dirac equation can be found in [R. J. Hughes, V. A. Kostelecký and M. M. Nieto, *Phys. Rev. D* (3) **34** (1986), no. 4, 1100–1107; [MR0852931](#); F. Correa and V. Jakubský, *Phys. Rev. D* **90** (2014), no. 12, 125003, [doi:10.1103/PhysRevD.90.125003](#)]. *E. Sadurní*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.