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Existence of non-topological multi-string solutions for a gravitational $O(3)$ gauge field model. (English summary)

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This interesting paper studies the Maxwell-Einstein field equations coupled to a vector field with SO(3) gauge invariance. The authors look for topological as well as non-topological solutions to the fully interacting system. This classification corresponds to (a) finite fields at infinity with an associated winding number or (b) non-finite (either divergent or vanishing) fields after the application of a stereographic projection. The existence of so-called cosmic strings without a quantized index in the presence of gravity stands out as an important result. See index J, type I, on page 1433.

In the first section of the paper, heuristic arguments in the discussion of field asymptotics are provided. This is done by a static analysis of the equations, similar to some techniques employed in soliton theory. A nice explanation of the method can be found in [R. Rajaraman, *Solitons and instantons*, North-Holland, Amsterdam, 1982; MR0719693]. The second section proves rigorously the existence of three types of solutions; namely, topological solutions and non-topological solutions of types I and II, by means of a perturbed Liouville equation and the use of norm inequalities. Finally, the third section provides estimates for physically relevant quantities associated with string solutions, such as energy, magnetic flux and Gaussian curvature; a bounded string number and a vanishing anti-string number are assumed here.

The model in the present paper is defined through a Lagrangian density in an unnumbered expression previous to (1.1). The authors refer to the last term in that expression as a Higgs potential. However, that expression does not really correspond to a ϕ-4 theory. In fact, the minima in this case are characterized by the relation $\phi \cdot n = \tau$, i.e., the corresponding manifold is a plane, whereas the minima in a quartic Higgs potential are characterized by $\phi^2 = 1$ for real ϕ; therefore the manifold is a sphere [see, e.g., M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley, Reading, MA, 1995 (pp. 694–695); MR1402248]. It is only after the stereographic projection is performed that a quartic dependence becomes apparent, with some additional non-linearities. The reader may want to contrast this model with the typical ferromagnet system studied in Section 3.3 of [R. Rajaraman, op. cit.].

{Reviewer's remarks: The authors point out the distinctive effects of gravity in their results. For historical reasons, it would be convenient to recall a classical work on the subject [J. A. Wheeler, Phys. Rev. (2) **97** (1955), 511–536; MR0067622], where localized structures emerge from the sole interaction between Maxwell fields and gravity. It seems that the present theory is only a few steps away from this situation, perhaps in the limit of small coupling between electromagnetism and the SO(3) gauge field.} *E. Sadurní*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.