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Gallone, Matteo (I-SISSA-NDM); **Michelangeli, Alessandro** (I-SISSA-NDM)
Hydrogenoid spectra with central perturbations. (English summary)
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As the authors put it aptly in the abstract of their paper, a Kreĭn-Vishik-Birman (KVB) scheme of self-adjoint extensions is utilized to solve the famous problem of a 3D delta potential located at the center of force in hydrogenoid atoms. This problem is well known to physicists, as it contains singularities when treated by means of Green's functions. The authors duly note that the point-like interaction in question emerges from the first corrections produced by a Foldy-Wouthuysen transformation applied to the Dirac Hamiltonian—specifically the so-called Darwin term. In essence, the delta distribution is replaced by a monoparametric class of boundary conditions at the origin of the radial coordinate, and only for s-waves. The main claim here is that the KVB scheme (based on quadratic forms) is superior in clarity to the von Neumann extension scheme (based on the Cayley transform). The Kreĭn ($\alpha = 0$) and Friedrichs ($\alpha = \infty$) extensions are recovered within the family of solutions, and the spectrum is shown to cover the real line as α takes all real values including infinity. The authors also provide a transcendental equation for the spectrum and explicit expansions for the wavefunctions, in agreement with [S. A. Albeverio et al., Ann. Inst. H. Poincaré Sect. A (N.S.) **38** (1983), no. 3, 263–293; MR0708965; W. Bulla and F. Gesztesy, J. Math. Phys. **26** (1985), no. 10, 2520–2528; MR0803795]. As a bonus, a threshold of the extension parameter for the appearance of bound states is reported in the case of repulsive Coulomb potentials—the threshold is hard to find elsewhere. Some of the technically demanding calculations in this paper can also be found in [S. A. Albeverio et al., op. cit.; W. Bulla and F. Gesztesy, op. cit.] regarding wavefunction asymptotics.

{Comments for the physicist: The peculiarities of this 'core' potential were historically considered by E. Fermi [Ric. Sci. **2** (1936), 13–52] and then by H. A. Bethe [Phys. Rev. **76** (1949), no. 1, 38–50, doi:10.1103/PhysRev.76.38] in nuclear physics. Its form reflects the cumbersome short-distance behavior of electrodynamics and it is typically treated by physicists in a regularization scheme (Fermi pseudopotentials) for 2D and 3D, as indicated in [K. Wódkiewicz, Phys. Rev. A (3) **43** (1991), no. 1, 68–76; MR1092320; T. T. Lê et al., Phys. Scr. **94** (2019), no. 6, 065203, doi:10.1088/1402-4896/ab0811]. It is not obvious how the old calculations could be reduced to redefinitions of boundary conditions at the origin, but one may note that the KVB parameter α is related to the intensity of renormalized delta potentials, simply by inspecting the form of the Green's function [C. Grosche, in *International Workshop Symmetry Methods in Physics: in memory of Professor Ya. A. Smorodinsky. Vol. 1*, 129–139, Joint Inst. Nucl. Res., Dubna, 1994]. A concise review of the KVB scheme in general can be found in [A. Alonso and B. Simon, J. Operator Theory **4** (1980), no. 2, 251–270; MR0595414].} *E. Sadurní*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.