Kiss Precise & Precise Quantization

Reincarnation of Apollonian Gaskets in a Quantum Fractal

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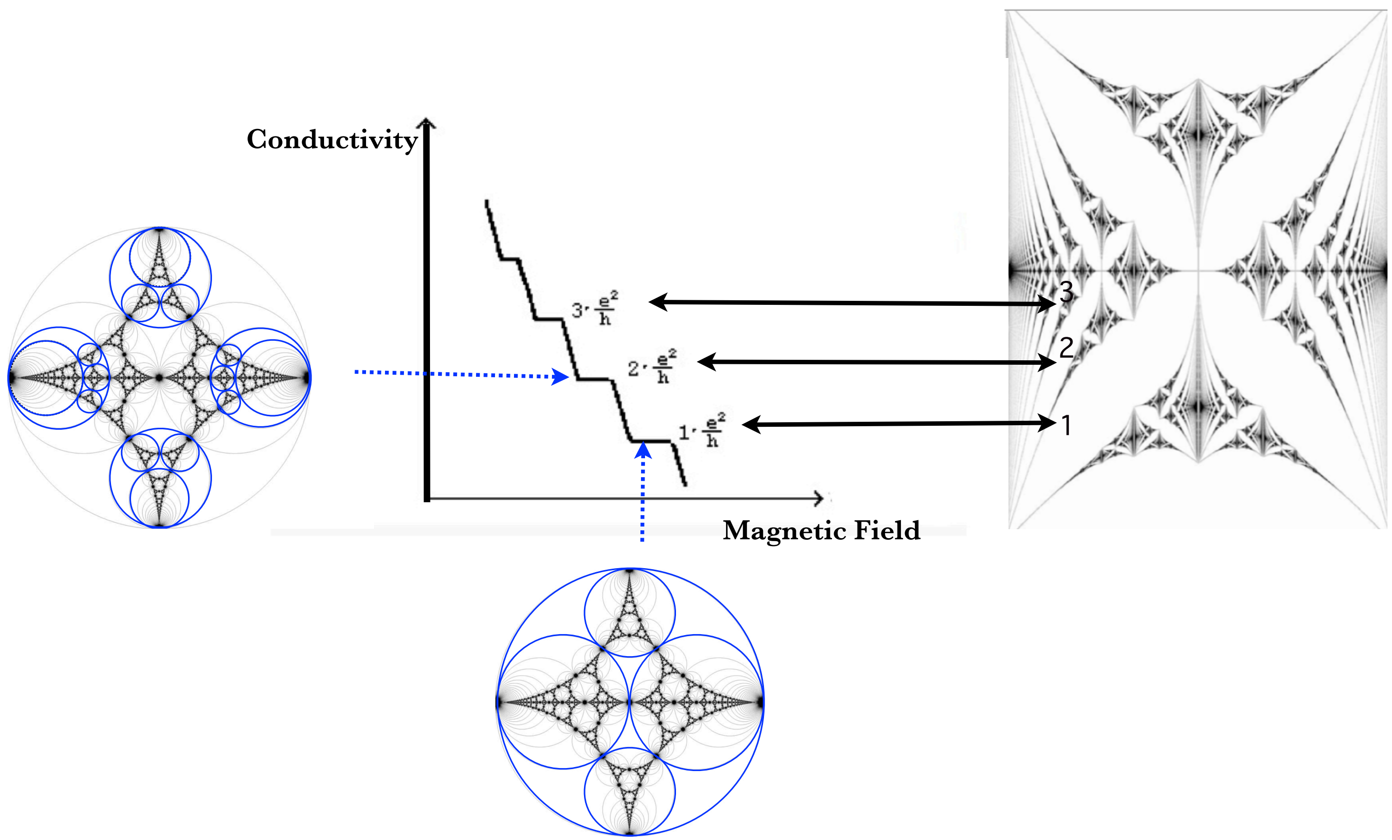
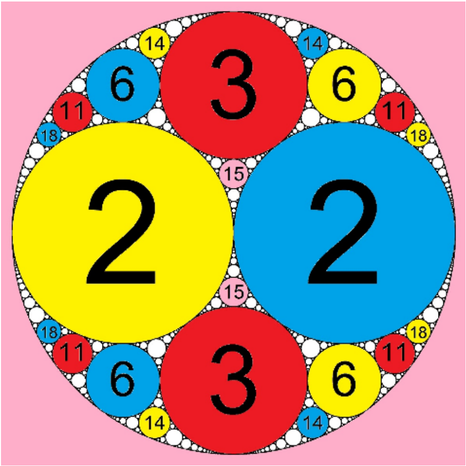
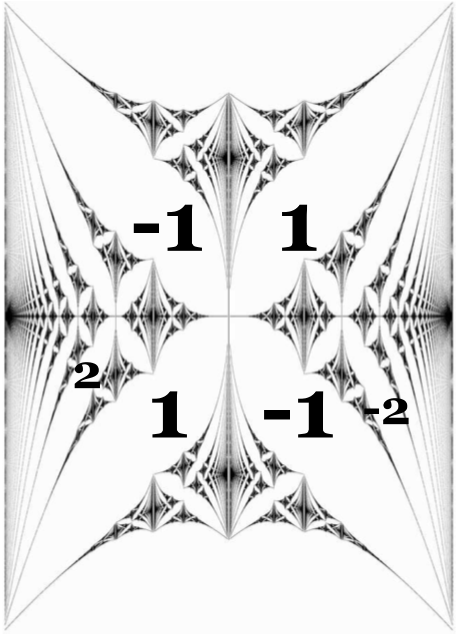


Tale of Two Fractals



Apollonian Gasket
Abstract Mathematics
Integers are curvatures of the circles

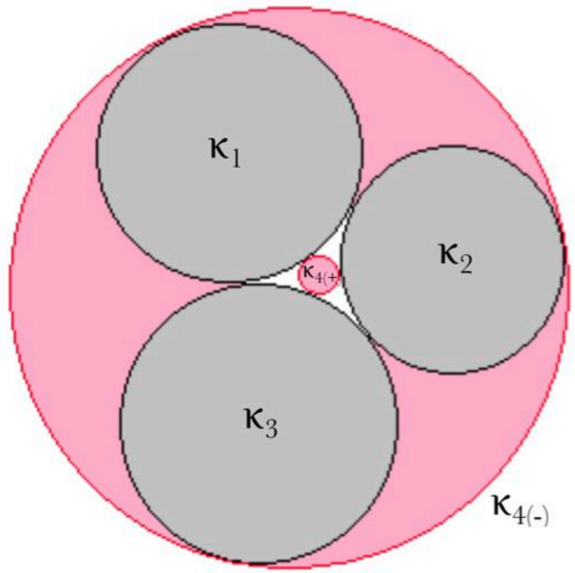


Hofstadter Butterfly: quantum factor describes real physical phenomena..
Quantum Hall effect
Integers (also known as Chern numbers) are physically measurbale quantitities

Tale of Two Precisions

- Kiss Precise
- Precise Quantization

Given three circles, how does one draw a fourth circle that is exactly tangent to all three?



Descartes circle theorem, 1643

Theorem (in a letter to Princess Elisabeth of Bohemia)

$$(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) = \frac{1}{2}(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2.$$

Fredrick Soddy, Nature, 1911

The Kiss Precise

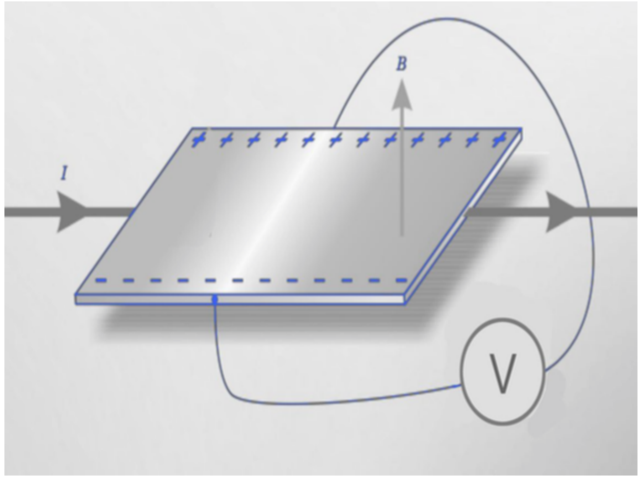
For pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

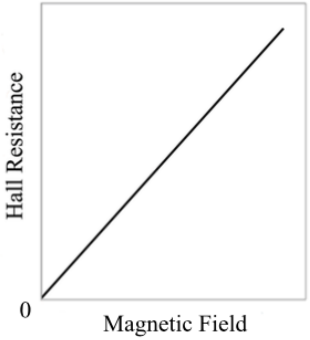
Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Artistic rendition
Sarah DeBauge, GMU

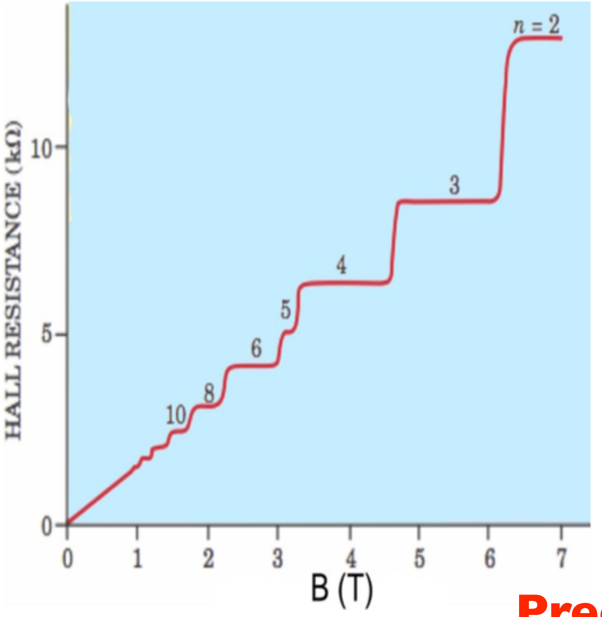
$$(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) = \frac{1}{2}(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2.$$

Edwin Hall
Classical Hall Effect
1878





Klaus Klitzing
Quantum Hall Effect, 1980,
Nobel prize 1985



$$\sigma_H = n\frac{e^2}{h}$$

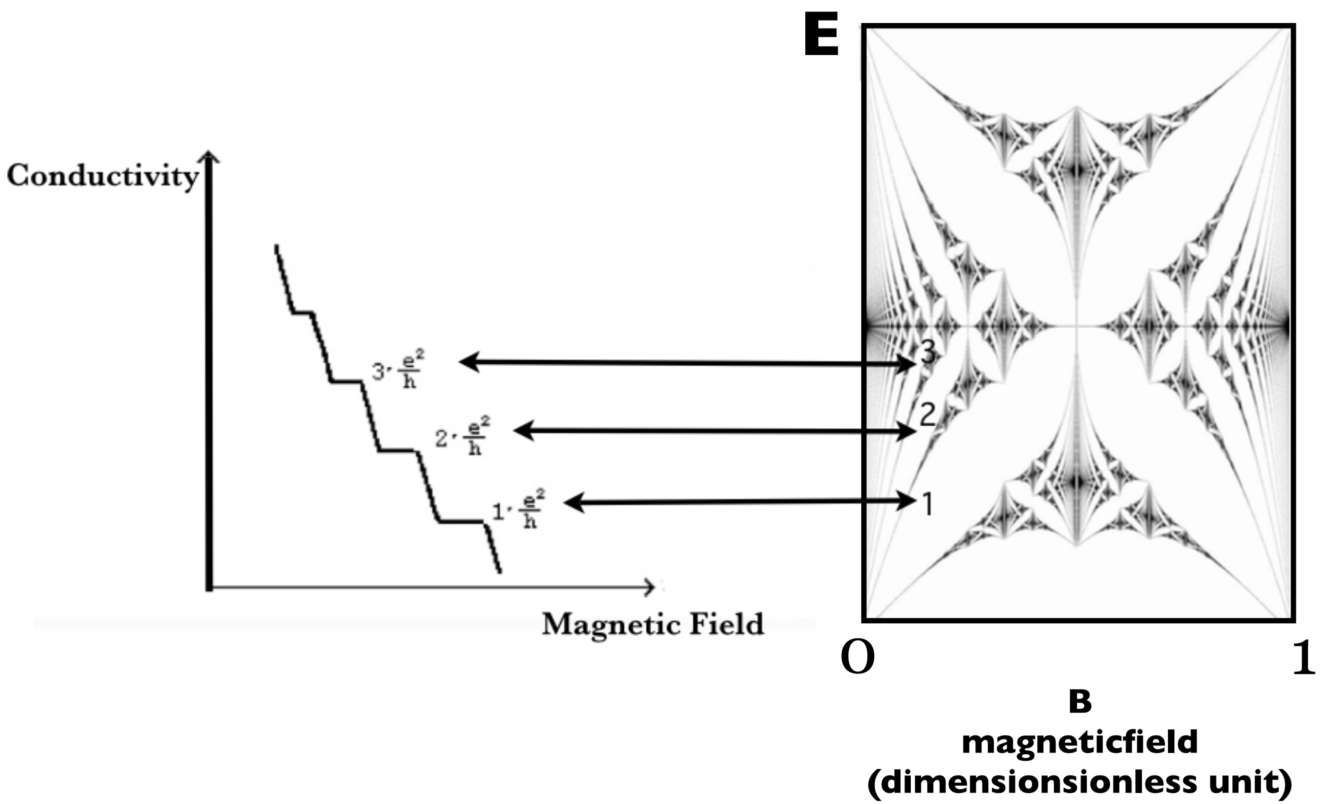
Samples: Two dimensional Insulators
Dirty samples & arbitrary Geometry

Related to Geometry of some abstract
(Hilbert Space)

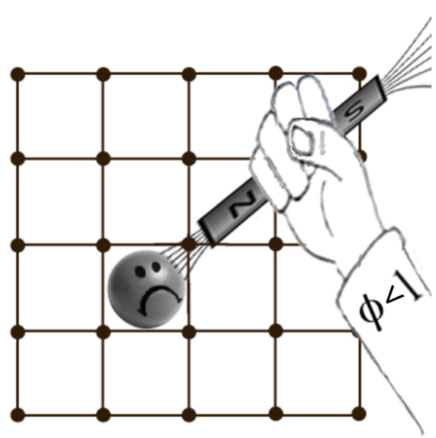
Precise Quantization:
Precision: One part in billion
Quantization has topological origin
(Berry phase: quantum anlog of classical parallel transport)

1975
before fractals...
before Quantum Hall

Hofstadter Butterfly simple model for Quantum Hall



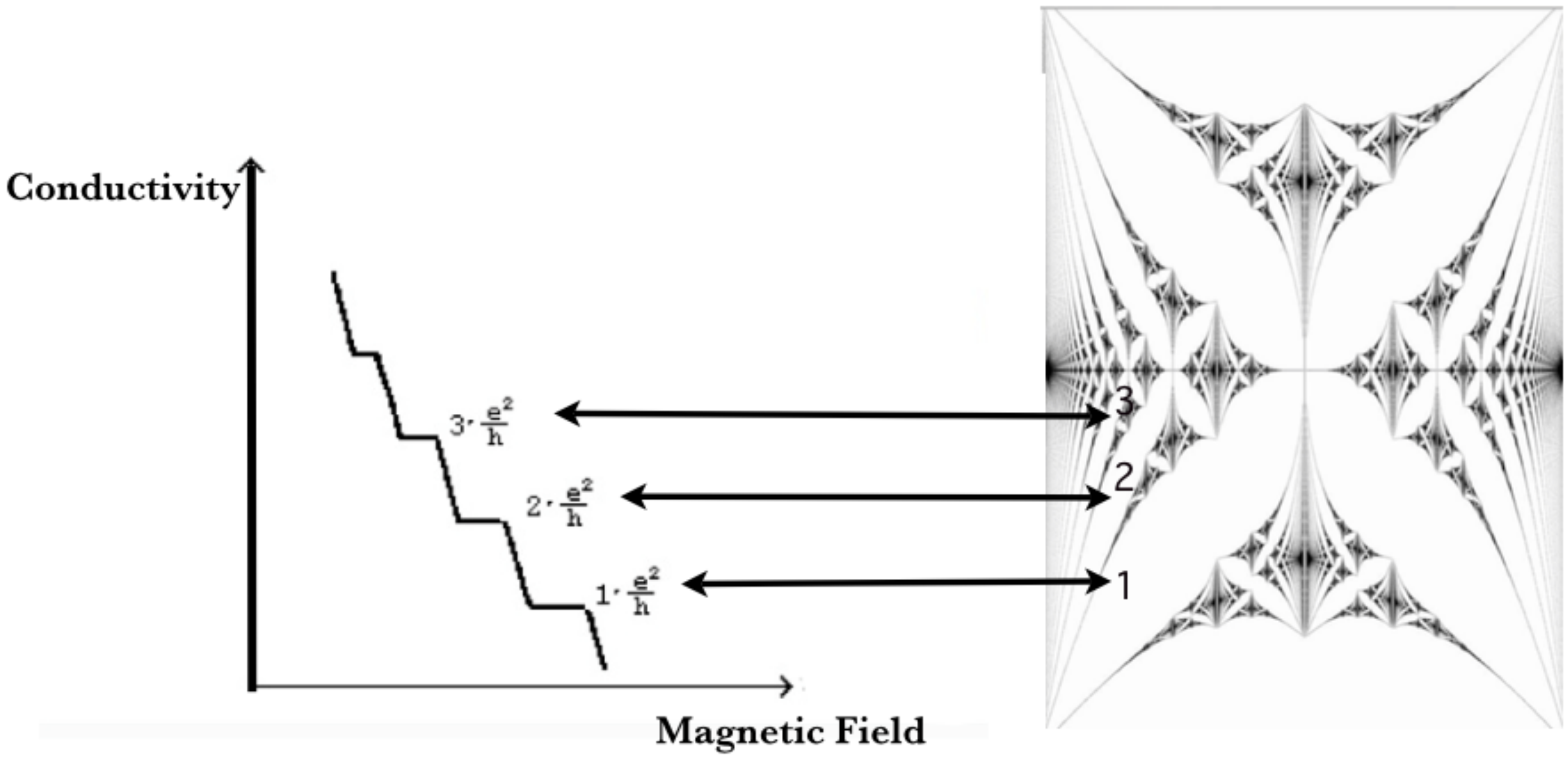
Empty regions (gaps) are forbidden energies
&
dark regions are allowed energies



Frustraed system

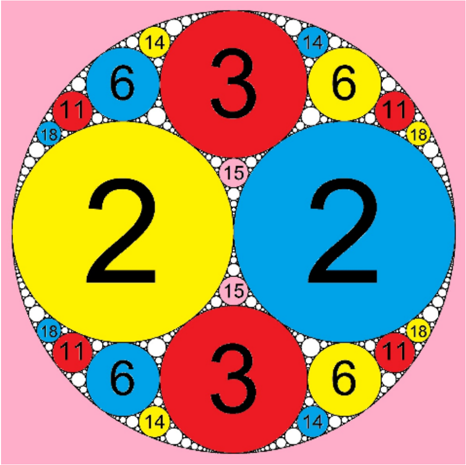
Hofstadter Butterfly is a multifractal made up of Integers
These integers are physically measurable quantities

Butterfly is a simplest theoretical model for Quantum Hall



Gaps of the butterfly are labeled by integers that represent quantum numbers of Hall conductivity

Another Fractal, made up of Integers ``Integral Apollonian"



$$(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) = \frac{1}{2}(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2.$$

If first four circles have integer curvatures, so are the rest

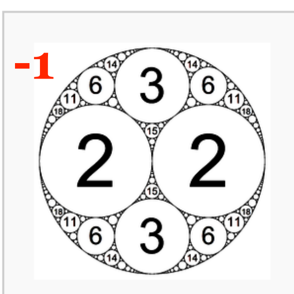
Integral Apollonian gaskets

Beginning curvatures	Symmetry
−1, 2, 2, 3, 3	D_2
−2, 3, 6, 7, 7	D_1
−3, 4, 12, 13, 13	D_1
−3, 5, 8, 8, 12	D_1
−4, 5, 20, 21, 21	D_1
−4, 8, 9, 9, 17	D_1
−5, 6, 30, 31, 31	D_1
−5, 7, 18, 18, 22	D_1
−6, 7, 42, 43, 43	D_1
−6, 10, 15, 19, 19	D_1
−6, 11, 14, 15, 23	C_1
−7, 8, 56, 57, 57	D_1
−7, 9, 32, 32, 36	D_1
−7, 12, 17, 20, 24	C_1
−8, 9, 72, 73, 73	D_1
−8, 12, 25, 25, 33	D_1
−8, 13, 21, 24, 28	C_1
−9, 10, 90, 91, 91	D_1
−9, 11, 50, 50, 54	D_1
−9, 14, 26, 27, 35	C_1
−9, 18, 19, 22, 34	C_1
−10, 11, 110, 111, 111	D_1
−10, 14, 35, 39, 39	D_1
−10, 18, 23, 27, 35	C_1

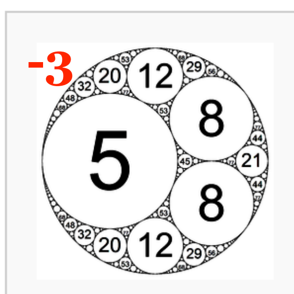
Integral Apollonian gaskets

Beginning curvatures	Symmetry
−11, 12, 132, 133, 133	D_1
−11, 13, 72, 72, 76	D_1
−11, 16, 36, 37, 45	C_1
−11, 21, 24, 28, 40	C_1
−12, 13, 156, 157, 157	D_1
−12, 16, 49, 49, 57	D_1
−12, 17, 41, 44, 48	C_1
−12, 21, 28, 37, 37	D_1
−12, 21, 29, 32, 44	C_1
−12, 25, 25, 28, 48	D_1
−13, 14, 182, 183, 183	D_1
−13, 15, 98, 98, 102	D_1
−13, 18, 47, 50, 54	C_1
−13, 23, 30, 38, 42	C_1
−14, 15, 210, 211, 211	D_1
−14, 18, 63, 67, 67	D_1
−14, 19, 54, 55, 63	C_1
−14, 22, 39, 43, 51	C_1
−14, 27, 31, 34, 54	C_1
−15, 16, 240, 241, 241	D_1
−15, 17, 128, 128, 132	D_1
−15, 24, 40, 49, 49	D_1
−15, 24, 41, 44, 56	C_1
−15, 28, 33, 40, 52	C_1

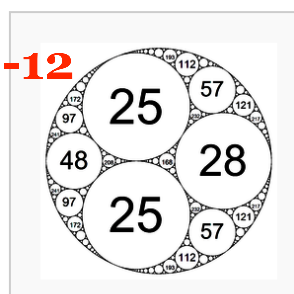
Note: Curvature of the outer bounding circle has to be taken with negative sign



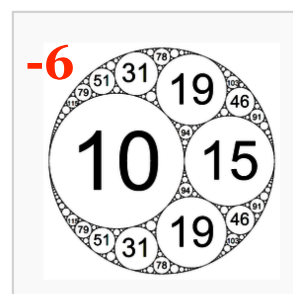
Integral Apollonian circle packing defined by circle curvatures of (−1, 2, 2, 3)



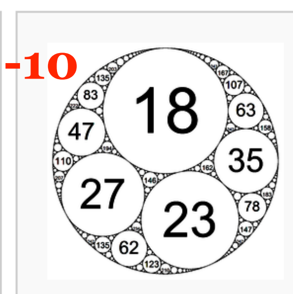
Integral Apollonian circle packing defined by circle curvatures of (−3, 5, 8, 8)



Integral Apollonian circle packing defined by circle curvatures of (−12, 25, 25, 28)



Integral Apollonian circle packing defined by circle curvatures of (−6, 10, 15, 19)

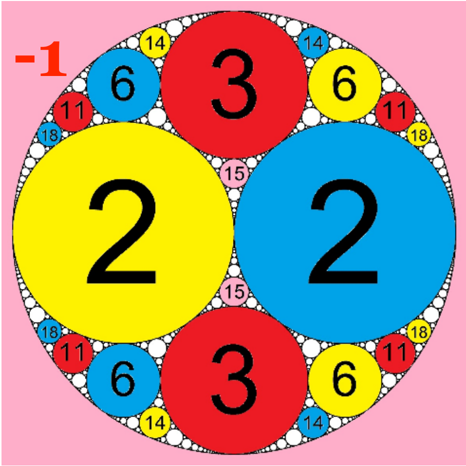


Integral Apollonian circle packing defined by circle curvatures of (−10, 18, 23, 27)

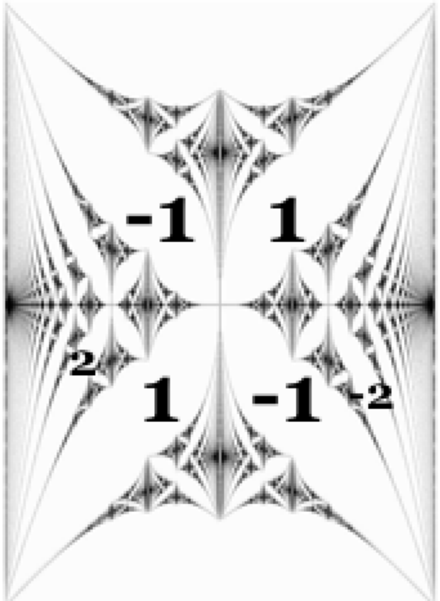
Integral Apollonian Gaskets & Butterfly Fractal are both made up of Integers

Could they possibly be related? ?????

Is this a marvelous example of a physical incarnation of abstract mathematics?



Integers are curvatures of
the circles



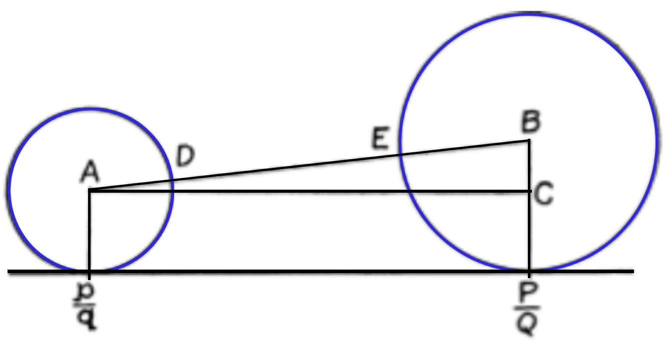
Integers are quantum numbers
of Hall conductivity

The two fractals are related--
integer curvatures determine the quantum numbers
Corresponding to Every Integral Apollonian, one can identify a
butterfly inside the butterfly fractal

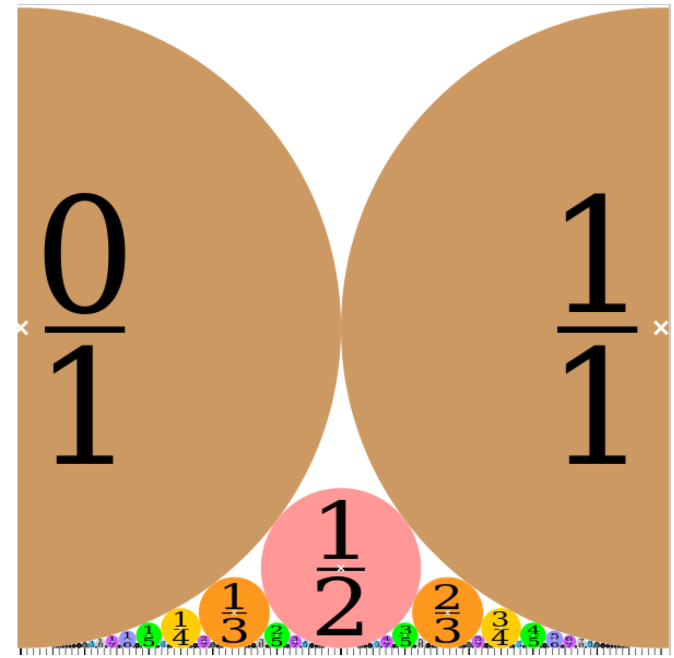
Apollolian Butterfly Connections (ABC)

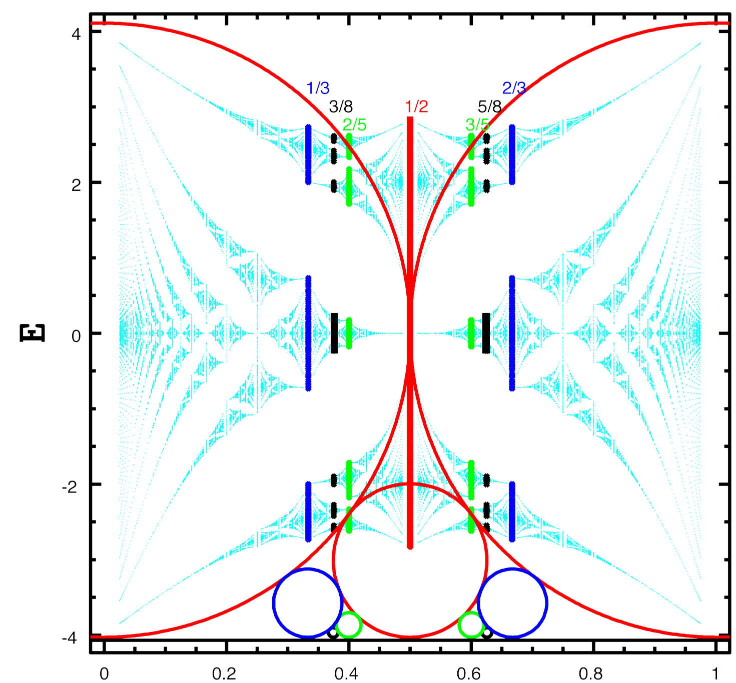
Ford Circles

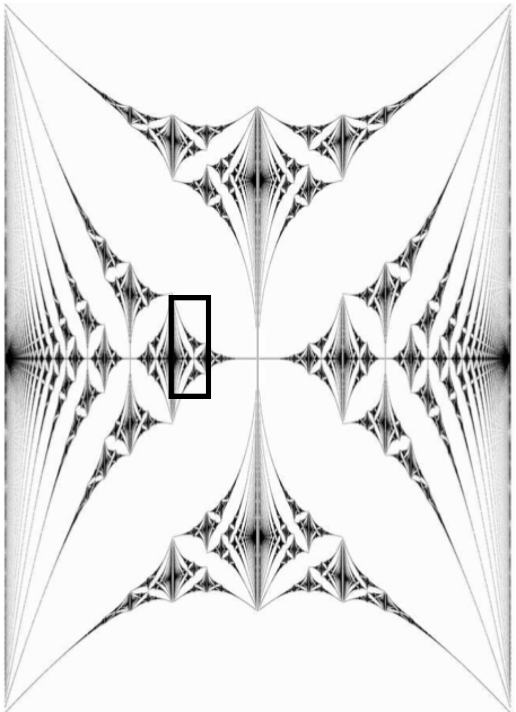
Consider x-axis, labelled by rationals. At each rational value (p/q), where p and q are relatively prime, draw a circle of radius 1/(2q^2), tangent to x-axis



- Two ford circles NEVER intersect
- If $|P q - pQ| > 1$, the two circles are external to each other.
- If $|P q - pQ| = 1$, the two circles kiss.
- If $|P q - pQ| = 0$, p/q=P/Q

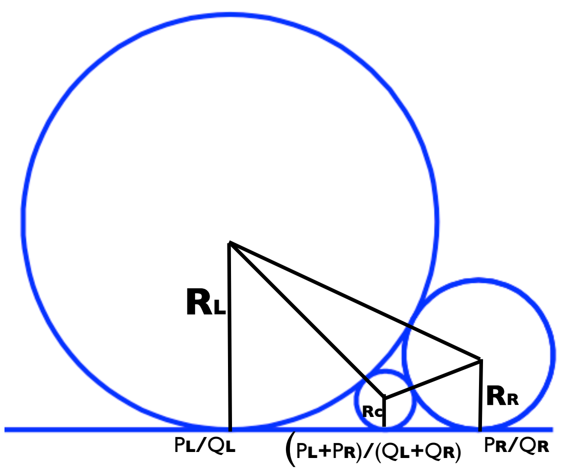


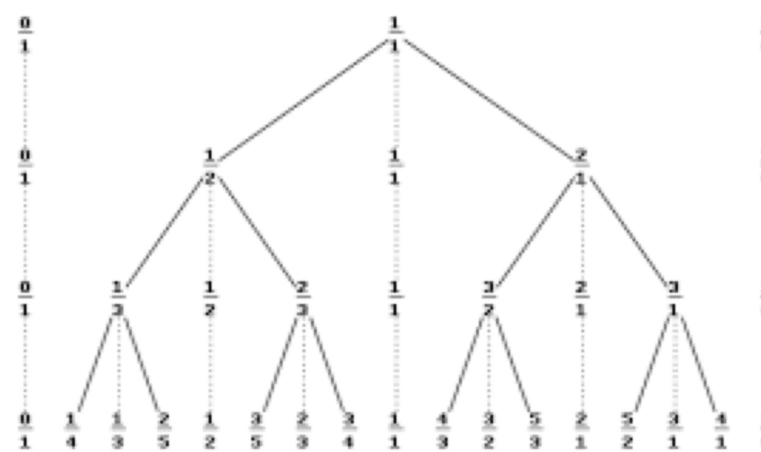




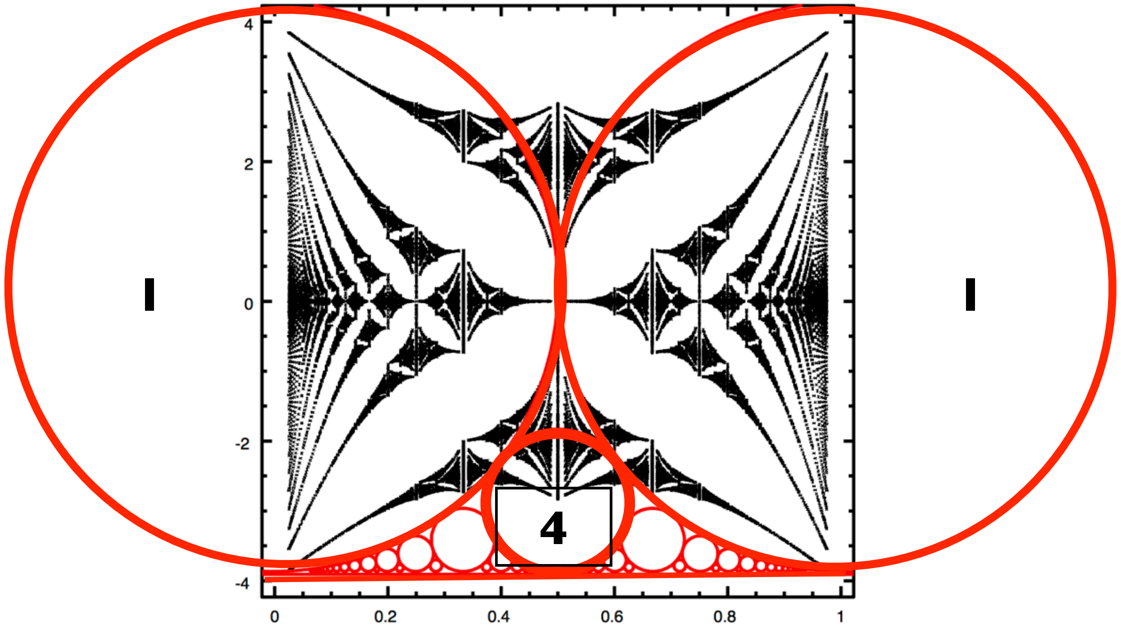
With every butterfly, associate three circles
(1) Left boundary of the butterfly
(2) Right Boundary of the Butterfly
(3) Center of the butterfly

It turns out that these three circles are mutually tangent
&
Three rationals satisfy Farey sum rule





Left, right boundaries, center & horizontal line satisfy Descartes's theorem



Butterfly is represented by Ford circles
Apollonian (4,1,1,0)

What Integral Apollonian does it represent

Integral Apollonian

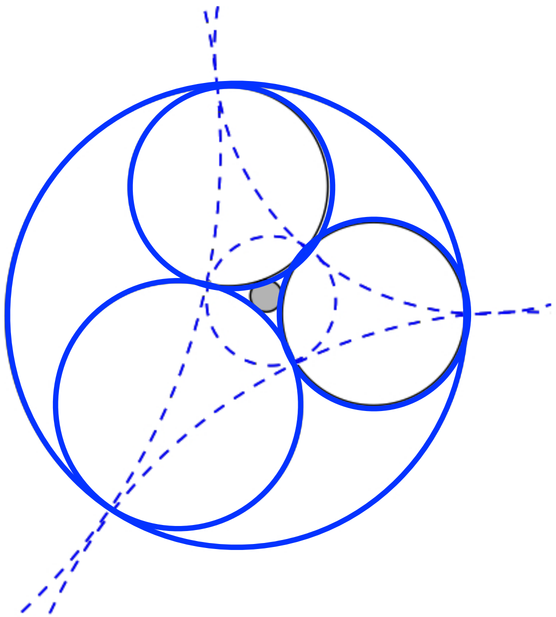
Beginning curvatures
−1, 2, 2, 3, 3
−2, 3, 6, 7, 7
−3, 4, 12, 13, 13
−3, 5, 8, 8, 12
−4, 5, 20, 21, 21
−4, 8, 9, 9, 17
−5, 6, 30, 31, 31
−5, 7, 18, 18, 22
−6, 7, 42, 43, 43
−6, 10, 15, 19, 19
−6, 11, 14, 15, 23
−7, 8, 56, 57, 57
−7, 9, 32, 32, 36
−7, 12, 17, 20, 24
−8, 9, 72, 73, 73
−8, 12, 25, 25, 33
−8, 13, 21, 24, 28
−9, 10, 90, 91, 91
−9, 11, 50, 50, 54
−9, 14, 26, 27, 35
−9, 18, 19, 22, 34
−10, 11, 110, 111, 111

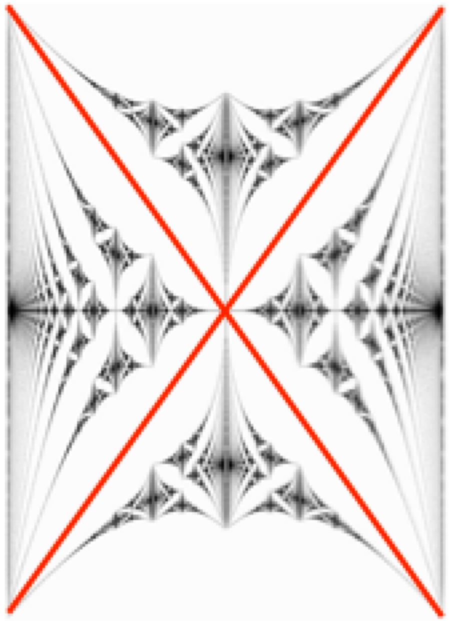
Duality Transformation

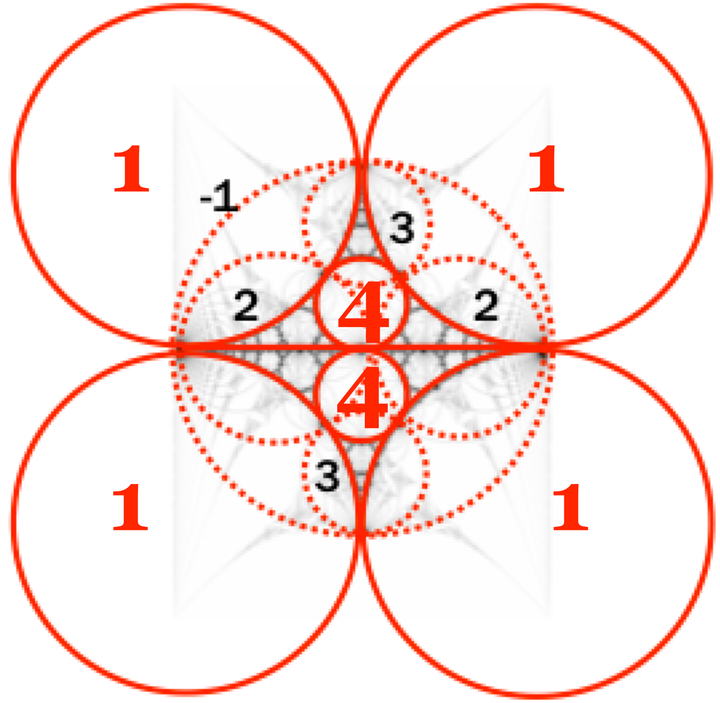
It turns out that the Ford-Apollonian is related to the $\mathcal{IAP}$ by a simple mathematical transformation, known as the *duality transformation*. Treating the curvatures of four kissing circles as a 4-vector, a duality transformation defined by a matrix $\hat{D}$ maps every Ford circle Apollonian A into its dual, $\bar{A}$ which can be identified with an $\mathcal{IAP}$,

$$\hat{D} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \tag{1}$$

$$\bar{A} = \hat{D}A \tag{2}$$

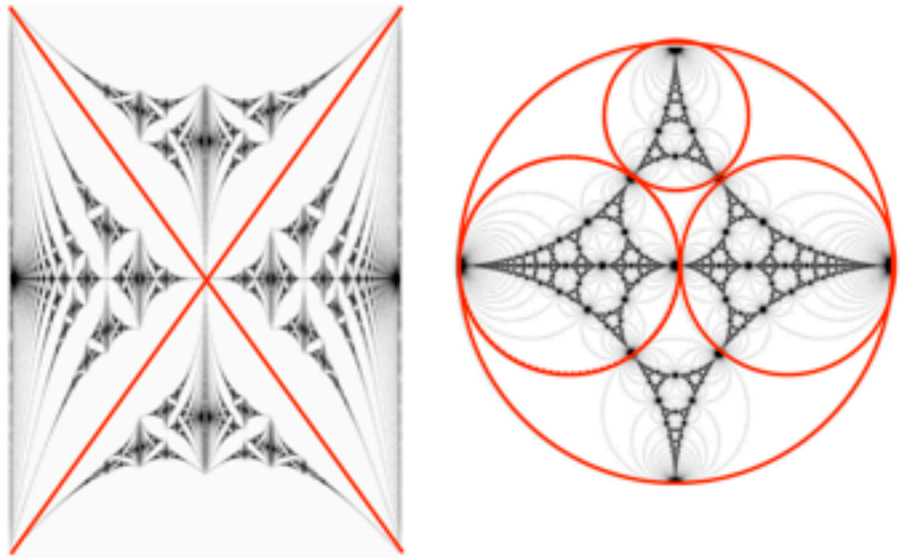




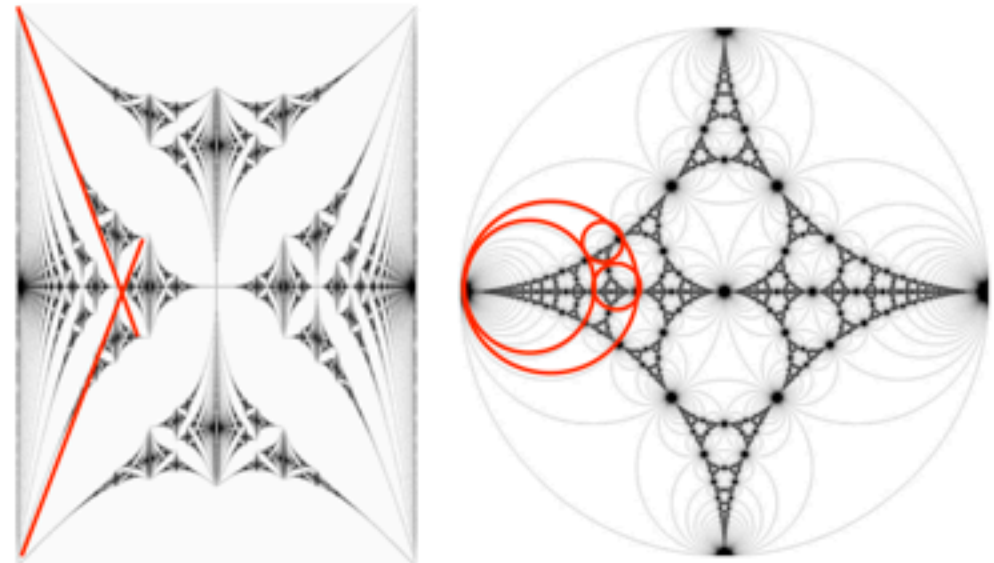


$$(4,1,1,0) \longleftrightarrow (-1,2,2,3)$$

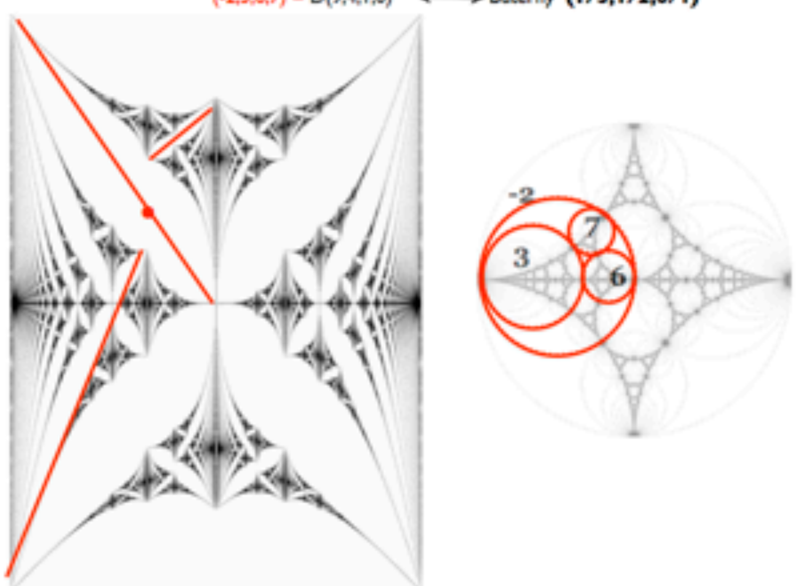
(-1,2,2,3) = D(4,1,1,0) ⟷ Buterfly **(1/2,1/1,0/1)**



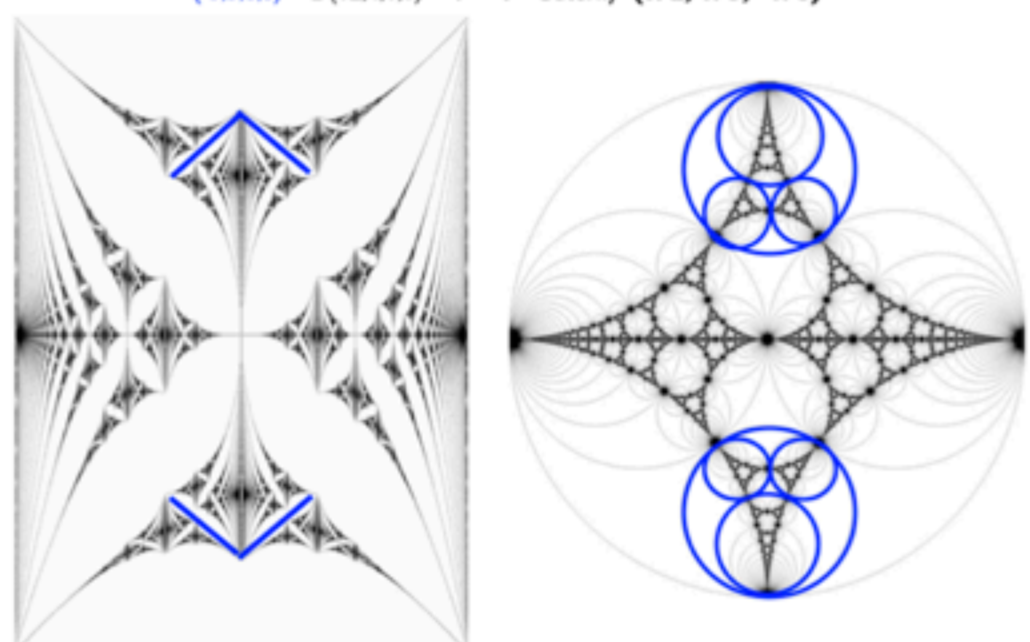
(-3,4,12,13) = D(9,4,1,0) ⟷ Buterfly **(1/4, 1/2, 0/1)**



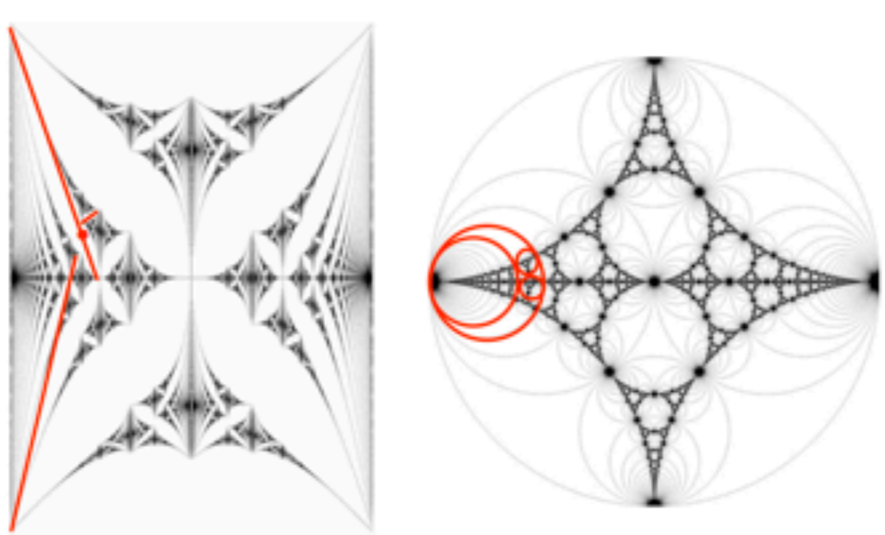
(-2,3,6,7) = D(9,4,1,0) ⟷ Buterfly **(1/3,1/2,0/1)**



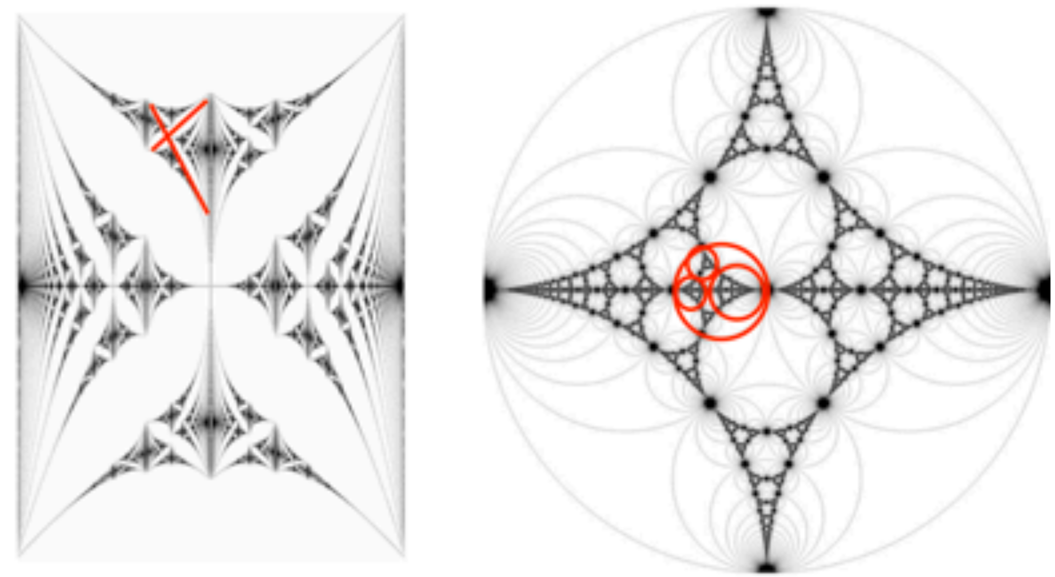
(-3,5,8,8) = D(12,4,1,1) ⟷ Buterfly **(1/2, 1/3, -1/3)**



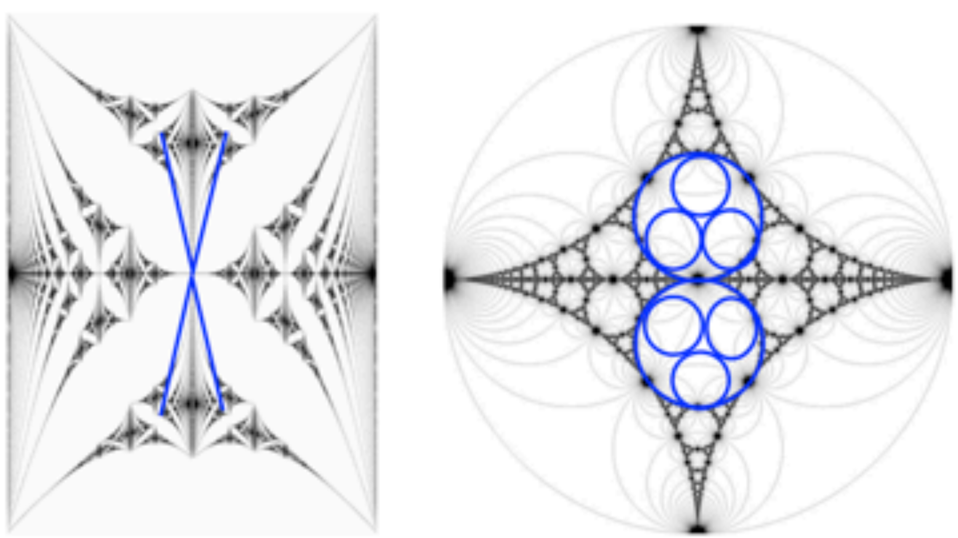
(-4,5,20,21) = D(25,16,1,0) ⟷ Buterfly **(1/5, 1/4, 0/1)**



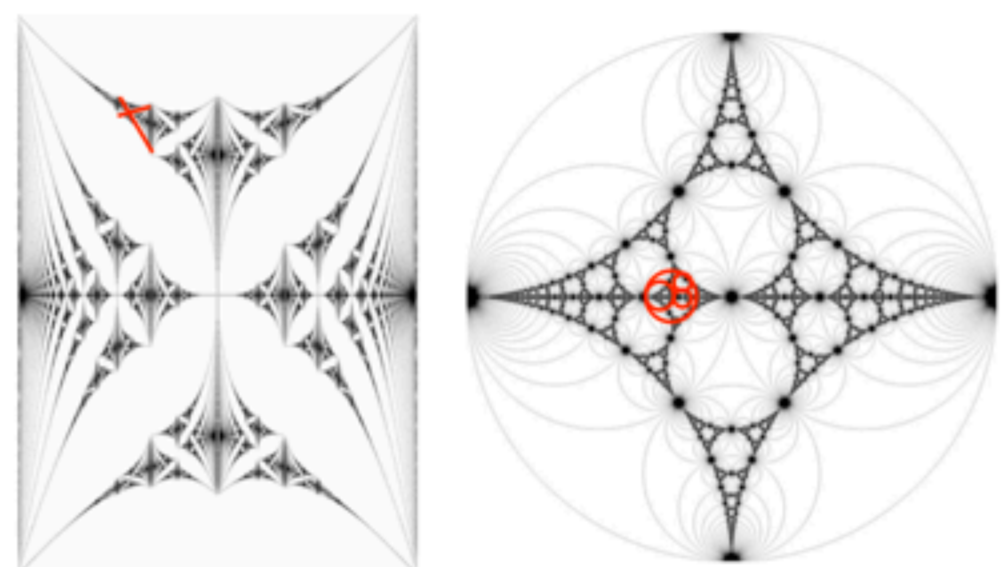
(-6,10,15,19) = D(25,9,4,0) ⟷ Buterfly **(2/5, 1/3,1/2)**

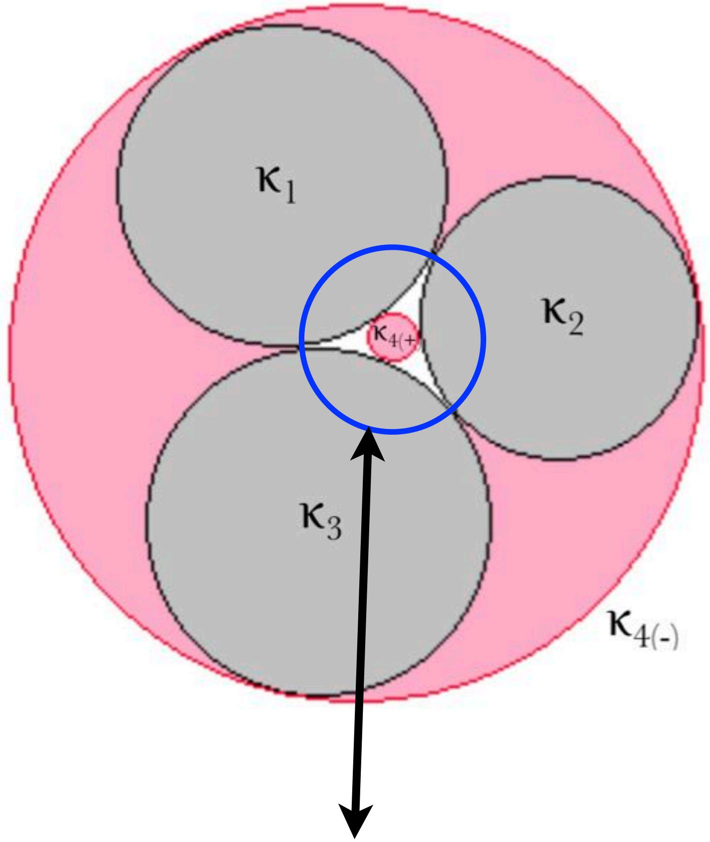


(-4,8,9,9) = D(15,3,2,2) ⟷ Buterfly (1/2, E=0)



(-12,21,28,37) = D(49,16,9,0) Buterfly **(2/7, 1/4,1/3)**

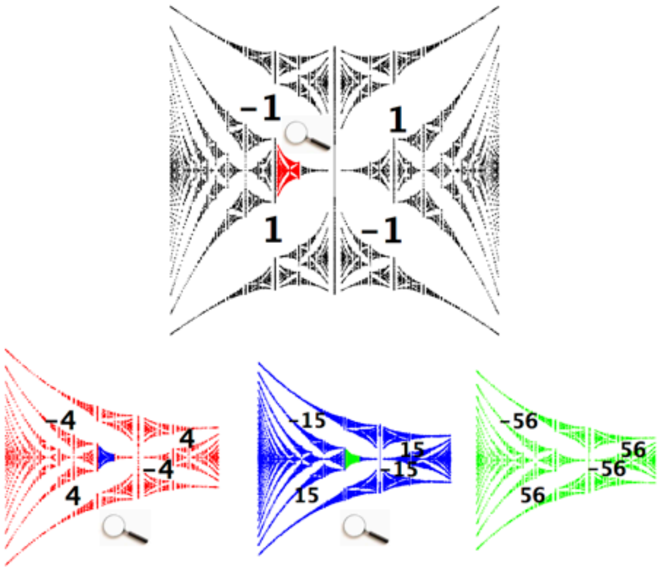


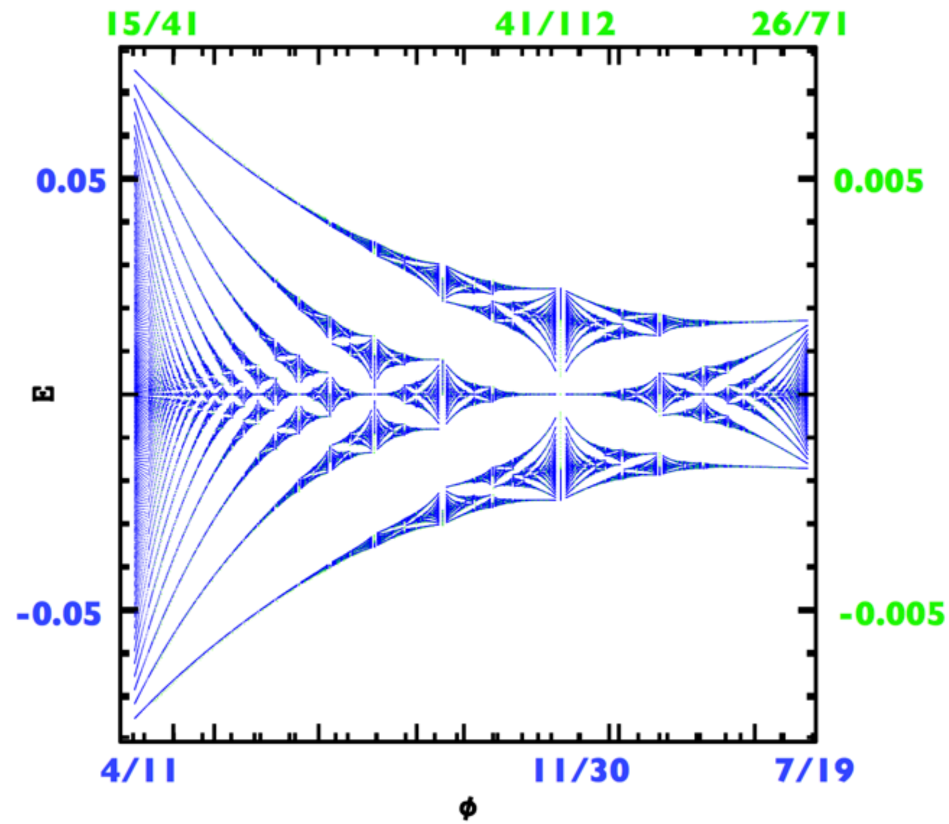


Topological Integer= $\pm\frac{\sqrt{\delta}}{2} = \pm\frac{1}{2}(\kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_1\kappa_3)^{1/4}$

$\pm$ 1/4[Sqrt[K+-K-]

Butterfly Fractal is made up of Integers
&
is self-similar

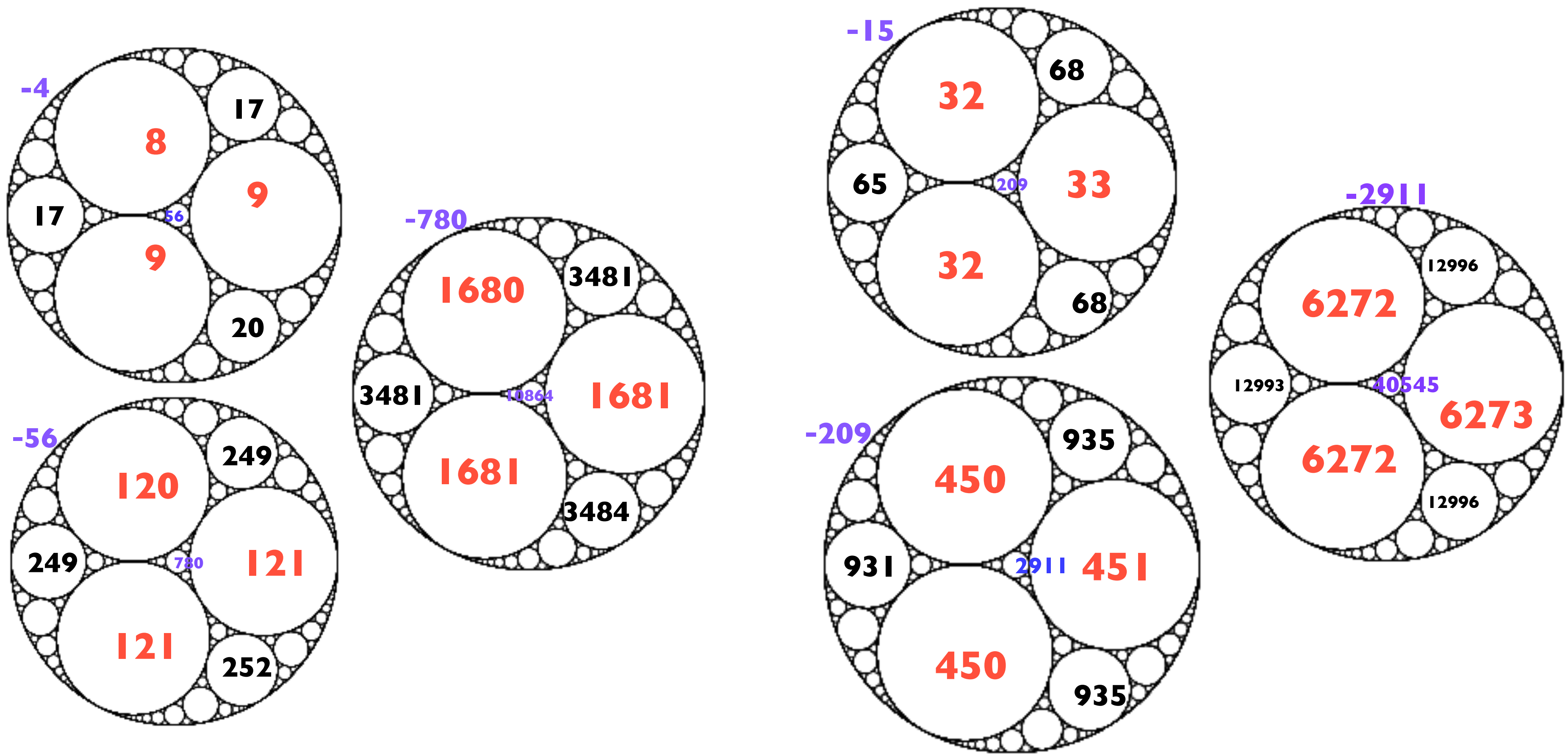




Universal Scaling: 4/1, 15/4, 56/15.... 2 + sqrt(3)

Size of the butterfly scales are [2+sqrt(3)]^2

Three-Fold Symmetry



Ratio of Outermost to Innermost curvatures
converge to the ratio [2+sqrt(3)]^2

Mysterious Three-Fold symmetry is hidden in the
butterfly as energy & flux scales go to zero

Why is this important

Quantum Hall describes what are known as Topological Insulators… Geometrical and Topological way to understand different states of matter.. what Einstein believed in..

There are other related states, such as quantum spin Hall and Fractional Quantum Hall.

Open Questions

- *Rigorus mathematical framework underlying ABC ??*
- *What do Topological Quantum Numbers mean for the Apollonian... Do Apollonians have a topological invariant that we can interpret geometrically ???*
- *Emerengence of 2+sqrt[3]... Is there some deeper significance of that*
- *Deeper understanding of the three-Fold symmetry at small energy and magnetic flux scale*
- *Do Apollolians "encode" other topological states of matter such as fractional quantum Hall and Quantum Spin Hall*
- *Butterfly fractal is also related to Mandelbroit set....and this relationship needs to be explored*
-

- *Book --" Butterfly in a Quantum World-Story of a most fascinating Quantum Fractal by Indu Satija: IOP book (coming soon)*