Continuation of bifurcations of cycles in dissipative PDEs

J. Sánchez and M. Net

Departament de Física Aplicada
Universitat Politècnica de Catalunya

Continuation of bifurcation curves

Consider an autonomous system of ODEs

$$\dot{y} = f(y,p), \quad (y,p) \in \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^2,$$

depending on two parameters $p = (p_1, p_2)$ obtained after spatial discretization of a system of parabolic PDEs ($n >> 1$).

Let $y(t) = \varphi(t, x, p)$ be its solution with initial condition $y(0) = x$ at $t = 0$ and for a fixed p.

We are interested in tracking curves of codimension-one bifurcations of periodic orbits in system with or without symmetries.

Let assume a matrix-free continuation code based on Newton-Krylov methods is available to follow the curves of solutions of

$$H(X) = 0$$

with $X \in \mathcal{U} \subset \mathbb{R}^{m+1}$ and $H(X) \in \mathbb{R}^m$, which requires the user to provide an initial solution X_0, and two subroutines:

- $\mathtt{fun}(\mathtt{X}, \mathtt{h})$ which computes $h = H(X)$ from X, and
- $\mathtt{dfun}(\mathtt{X}, \delta\mathtt{X}, \delta\mathtt{h})$ which computes $\delta h = D_X H(X)\delta X$ from X, and δX.

Saddle–node and period doubling bifurcations

The saddle-node ($\lambda = 1$) and period doubling ($\lambda = -1$) bifurcations of periodic orbits are solutions of the system $H(x, u, T, p) = 0$ given by

$$\begin{aligned} x - \varphi(T, x, p) &= 0, \\ g(x) &= 0, \\ \lambda u - \left(D_x\varphi(T, x, p)u - \frac{1}{2}(1+\lambda)\frac{\langle f, u\rangle}{\langle f, f\rangle} f \right) &= 0, \\ \langle u_r, u\rangle &= 1. \end{aligned}$$

- $g(x) = 0$ is a phase condition to select a single point on the periodic orbit. We use $g(x) = \langle v_\pi, x - x^{(\pi)}\rangle = 0$.
- $f = f(x, p)$ is the vector field evaluated at (x, p).
- $\langle u_r, u\rangle = 1$ fixes the indetermined constant of the eigenvalue problem, u_r being a reference vector. We use $u_r = u$.
- The last term of the third equation is Wieland's deflation, which guarantees the regularity of the system by shifting the $+1$ multiplier associated with $f(x, p)$ to zero.

$X = (x, u, T, p)$ has dimension $2n + 3$, and the $2n + 2$ equations define the curve of solutions.

In order to compute $H(x, u, T, p)$, we define

$$y(t) = \varphi(t, x, p)$$
$$y_1(t) = D_x\varphi(t, x, p)u$$

and, taking into account that

$$D_tD_x\varphi(t, x, p) = D_yf(\varphi(t, x, p), p)D_x\varphi(t, x, p), \text{ and } D_x\varphi(0, x, p) = I$$

the following system has to be integrated during a time T

$$\dot{y} = f(y, p), \qquad y(0) = x$$
$$\dot{y}_1 = D_yf(y, p)y_1, \qquad y_1(0) = u.$$

Then

$$\varphi(T, x, p) = y(T)$$
$$D_x\varphi(T, x, p)u = y_1(T).$$

The action of $D_X H(x,u,T,p)$ on $(\delta x, \delta u, \delta T, \delta p)$ is

$$
\begin{aligned}
&\delta x - D_t\varphi(T,x,p)\delta T - D_x\varphi(T,x,p)\delta x - D_p\varphi(T,x,p)\delta p,\\
&Dg(x)\delta x,\\
&\lambda\delta u - D^2_{tx}\varphi(T,x,p)(u,\delta T) - D^2_{xx}\varphi(T,x,p)(u,\delta x) - D^2_{xp}\varphi(T,x,p)(u,\delta p)\\
&\; - D_x\varphi(T,x,p)\delta u\\
&\; + \frac{1+\lambda}{2\langle w,w\rangle}\left(\langle w,u\rangle z + \left(\langle z,u\rangle + \langle w,\delta u\rangle - \frac{2\langle w,z\rangle}{\langle w,w\rangle}\langle w,u\rangle\right) w\right),\\
&\langle u_r, \delta u\rangle,
\end{aligned}
$$

where $w = f(x,p)$ and $z = D_y f(x,p)\delta x + D_p f(x,p)\delta p$. Lets define

$$
\begin{aligned}
y(t) &= \varphi(t,x,p),\\
y_1(t) &= D_x\varphi(t,x,p)u,\\
y_2(t) &= D_x\varphi(t,x,p)\delta x + D_p\varphi(t,x,p)\delta p,\\
y_3(t) &= D^2_{xx}\varphi(t,x,p)(u,\delta x) + D^2_{xp}\varphi(t,x,p)(u,\delta p),\\
y_4(t) &= D_x\varphi(t,x,p)\delta u.
\end{aligned}
$$

If

$$\begin{aligned} y(t) &= \varphi(t, x, p), \\ y_1(t) &= D_x\varphi(t, x, p)u, \\ y_2(t) &= D_x\varphi(t, x, p)\delta x + D_p\varphi(t, x, p)\delta p, \\ y_3(t) &= D^2_{xx}\varphi(t, x, p)(u, \delta x) + D^2_{xp}\varphi(t, x, p)(u, \delta p), \\ y_4(t) &= D_x\varphi(t, x, p)\delta u, \end{aligned}$$

the system which must be integrated to obtain $y(T)$, $y_i(T)$, $i = 1, \ldots, 4$ is

$$\begin{aligned} \dot{y} &= f(y, p), & y(0) &= x \\ \dot{y}_1 &= D_y f(y, p)y_1, & y_1(0) &= u \\ \dot{y}_2 &= D_y f(y, p)y_2 + D_p f(y, p)\delta p, & y_2(0) &= \delta x \\ \dot{y}_3 &= D_y f(y, p)y_3 + D^2_{yy} f(y, p)(y_1, y_2) + D^2_{yp} f(y, p)(y_1, \delta p), & y_3(0) &= 0 \\ \dot{y}_4 &= D_y f(y, p)y_4, & y_4(0) &= \delta u. \end{aligned}$$

Neimark-Sacker bifurcations

The Hopf bifurcations of periodic orbits with multiplier $e^{i\theta}$ and eigenvector $u + iv$ are solutions of the system $H(x, u, v, T, \theta, p) = 0$ given by

$$x - \varphi(T, x, p) = 0,$$

$$g(x) = 0,$$

$$u \cos\theta - v \sin\theta - D_x\varphi(T, x, p)u = 0,$$

$$u \sin\theta + v \cos\theta - D_x\varphi(T, x, p)v = 0,$$

$$\langle u, u \rangle + \langle v, v \rangle = 1,$$

$$\langle u, v \rangle = 0.$$

- $g(x) = 0$ is the phase condition $g(x) = \langle v_\pi, x - x^{(\pi)} \rangle = 0$.
- The two last equations uniquely determine the eigenvector $u + iv$.

Now $X = (x, u, v, T, \theta, p)$ has dimension $3n + 4$, and the $3n + 3$ equations define the curve of solutions.

Pitchfork bifurcations

If the initial system is $\mathcal{T}$-invariant, $f(\mathcal{T}x, p) = \mathcal{T}f(x, p)$ with $\mathcal{T}^2 = I$, and $\mathcal{T}x = x$, the pitchfork bifurcation points of periodic orbits are solutions of the system $H(x, u, T, \xi, p) = 0$ are given by

$$\begin{aligned} x - \varphi(T, x, p) + \xi\phi &= 0, \\ g(x) &= 0, \\ \langle x, \phi \rangle &= 0, \\ u - \left(D_x\varphi(T, x, p)u - \frac{\langle f, u \rangle}{\langle f, f \rangle} f \right) &= 0, \\ \langle u_r, u \rangle &= 1. \end{aligned}$$

- The slack variable ξ and the third equation are introduced to make the system regular. Moreover $\xi = 0$ at the solution.
- $g(x) = 0$ is the phase condition $g(x) = \langle v_\pi, x - x^{(\pi)} \rangle = 0$.
- ϕ is a given antisymmetric vector, $\mathcal{T}\phi = -\phi$.
- The last equation uniquely determines the eigenvector u.

Now $X = (x, u, T, \xi, p)$ has dimension $2n + 4$, and the $2n + 3$ equations define the curve of solutions.

Thermal convection in binary fluid mixtures

The equations in $\Omega = [0, \Gamma] \times [0, 1]$ for the perturbation of the basic state ($\mathbf{v}_c = 0$, $T_c = T_c(0) - z$, and $C_c = C_c(0) - z$) in non-dimensional form are

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \sigma \Delta \mathbf{v} - \nabla p + \sigma Ra(\Theta + SC)\hat{e}_z,$$

$$\partial_t \Theta + (\mathbf{v} \cdot \nabla)\Theta = \Delta \Theta + v_z,$$

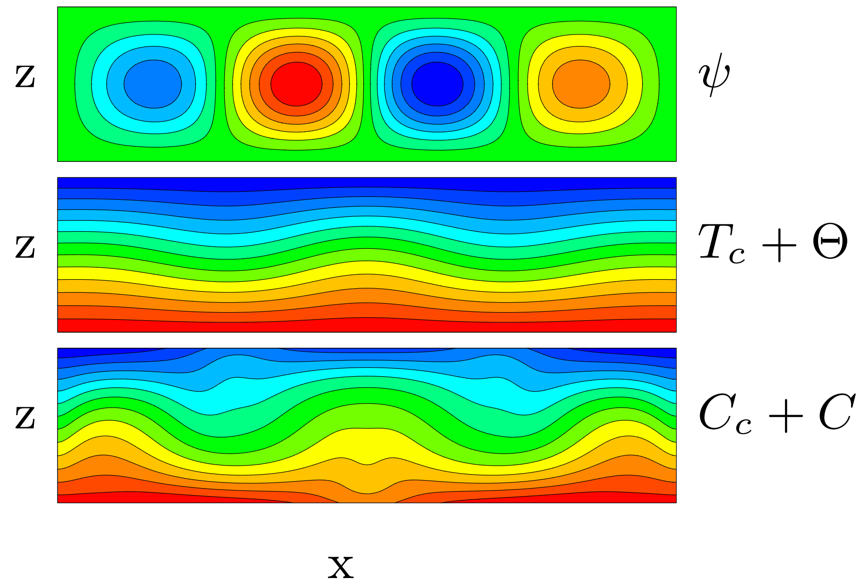
$$\partial_t C + (\mathbf{v} \cdot \nabla)C = L(\Delta C - \Delta \Theta) + v_z,$$

$$\nabla \cdot \mathbf{v} = 0.$$

The boundary conditions are non-slip for $\mathbf{v}$, constant temperatures at top and bottom and insulating lateral walls for $\Theta = T - T_c$, and impermeable boundaries for C.

The parameters are

Γ	Aspect ratio (4)
S	Separation ratio (-0.1)
L	Lewis number (0.03)
σ	Prandtl number (control)
Ra	Rayleigh number (control)



To simplify the system, a streamfunction $\mathbf{v} = (-\partial_z \psi, \partial_x \psi)$, and an auxiliary function $\eta = C - \Theta$ are used. Then

$$\begin{aligned}
\partial_t \Delta\psi + J(\psi, \Delta\psi) =& \sigma \Delta^2 \psi + \sigma Ra \left[(S+1) \partial_x \Theta + S \partial_x \eta \right], \\
\partial_t \Theta + J(\psi, \Theta) =& \Delta\Theta + \partial_x \psi, \\
\partial_t \eta + J(\psi, \eta) =& L \Delta\eta - \Delta\Theta,
\end{aligned}$$

with $J(f, g) = \partial_x f \partial_z g - \partial_z f \partial_x g$. The boundary conditions are now

$$\begin{aligned}
\psi = \partial_n \psi = \partial_n \eta = 0 \quad & \text{at} \quad \partial\Omega, \\
\Theta = 0 \quad & \text{at} \quad z = 0, 1, \\
\partial_x \Theta = 0 \quad & \text{at} \quad x = 0, \Gamma.
\end{aligned}$$

The symmetry group of the equations is $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the two reflections:

$$\begin{aligned}
R_x : (t, x, z, \psi, \Theta, \eta) \to (t, \Gamma - x, z, -\psi, \Theta, \eta), \\
R_z : (t, x, z, \psi, \Theta, \eta) \to (t, x, 1 - z, -\psi, -\Theta, -\eta).
\end{aligned}$$

Variational equations

$$\begin{aligned}
\partial_t \Delta\psi_1 + J(\psi, \Delta\psi_1) + J(\psi_1, \Delta\psi) =& \sigma\Delta^2\psi_1 + \sigma Ra\left[(S+1)\partial_x\Theta_1 + S\partial_x\eta_1\right],\\
\partial_t\Theta_1 + J(\psi, \Theta_1) + J(\psi_1, \Theta) =& \Delta\Theta_1 + \partial_x\psi_1,\\
\partial_t\eta_1 + J(\psi, \eta_1) + J(\psi_1, \eta) =& L\Delta\eta_1 - \Delta\Theta_1,
\end{aligned}$$

$$\begin{aligned}
\partial_t \Delta\psi_2 + J(\psi, \Delta\psi_2) + J(\psi_2, \Delta\psi) =& \sigma\Delta^2\psi_2 + \sigma Ra\left[(S+1)\partial_x\Theta_2 + S\partial_x\eta_2\right] + \delta\sigma\Delta^2\psi\\
&+ (\sigma\delta Ra + \delta\sigma Ra)\left[(S+1)\partial_x\Theta + S\partial_x\eta\right],\\
\partial_t\Theta_2 + J(\psi, \Theta_2) + J(\psi_2, \Theta) =& \Delta\Theta_2 + \partial_x\psi_2,\\
\partial_t\eta_2 + J(\psi, \eta_2) + J(\psi_2, \eta) =& L\Delta\eta_2 - \Delta\Theta_2,
\end{aligned}$$

$$\begin{aligned}
\partial_t \Delta\psi_3 + J(\psi, \Delta\psi_3) + J(\psi_3, \Delta\psi) =& \sigma\Delta^2\psi_3 + \sigma Ra\left[(S+1)\partial_x\Theta_3 + S\partial_x\eta_3\right] + \delta\sigma\Delta^2\psi_1\\
&+ (\sigma\delta Ra + \delta\sigma Ra)\left[(S+1)\partial_x\Theta_1 + S\partial_x\eta_1\right]\\
&- J(\psi_1, \Delta\psi_2) - J(\psi_2, \Delta\psi_1),\\
\partial_t\Theta_3 + J(\psi, \Theta_3) + J(\psi_3, \Theta) =& \Delta\Theta_3 + \partial_x\psi_3 - J(\psi_1, \Theta_2) - J(\psi_2, \Theta_1),\\
\partial_t\eta_3 + J(\psi, \eta_3) + J(\psi_3, \eta) =& L\Delta\eta_3 - \Delta\Theta_3 - J(\psi_1, \eta_2) - J(\psi_2, \eta_1).
\end{aligned}$$

Discretization

The functions ψ, Θ, and η are approximated by a pseudo-spectral method. Collocation on a mesh of $n_x \times n_z = 64 \times 16$ ($n = 3072$) Gauss-Lobatto points is used.

Higher resolutions have been used to check the results.

The stiff system of ODEs obtained can be written as

$$B\dot{u} = Lu + N(u)$$

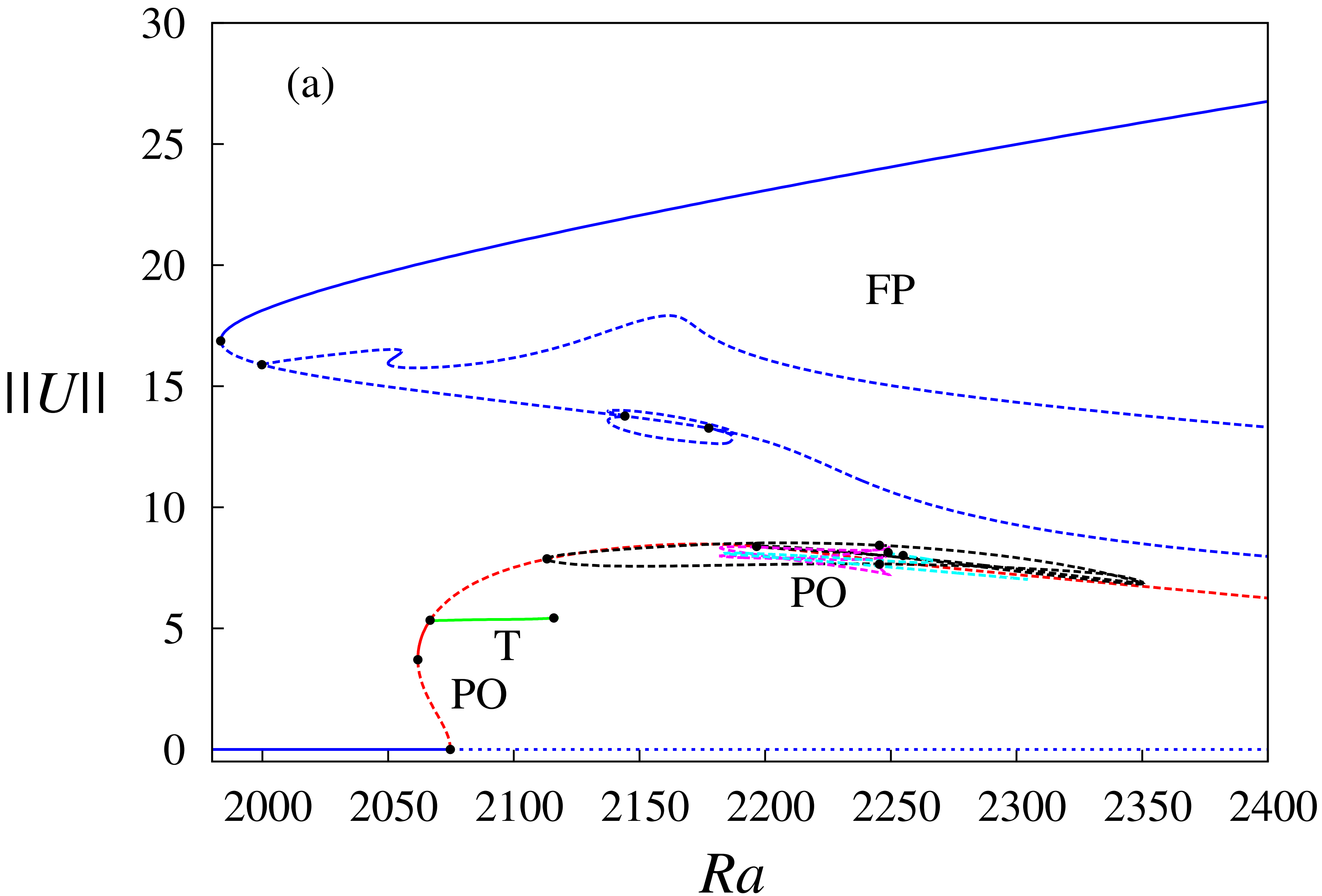
with $u = (\psi_{ij}, \Theta_{ij}, \eta_{ij})$.

It is integrated by using fifth-order BDF-extrapolation formulas:

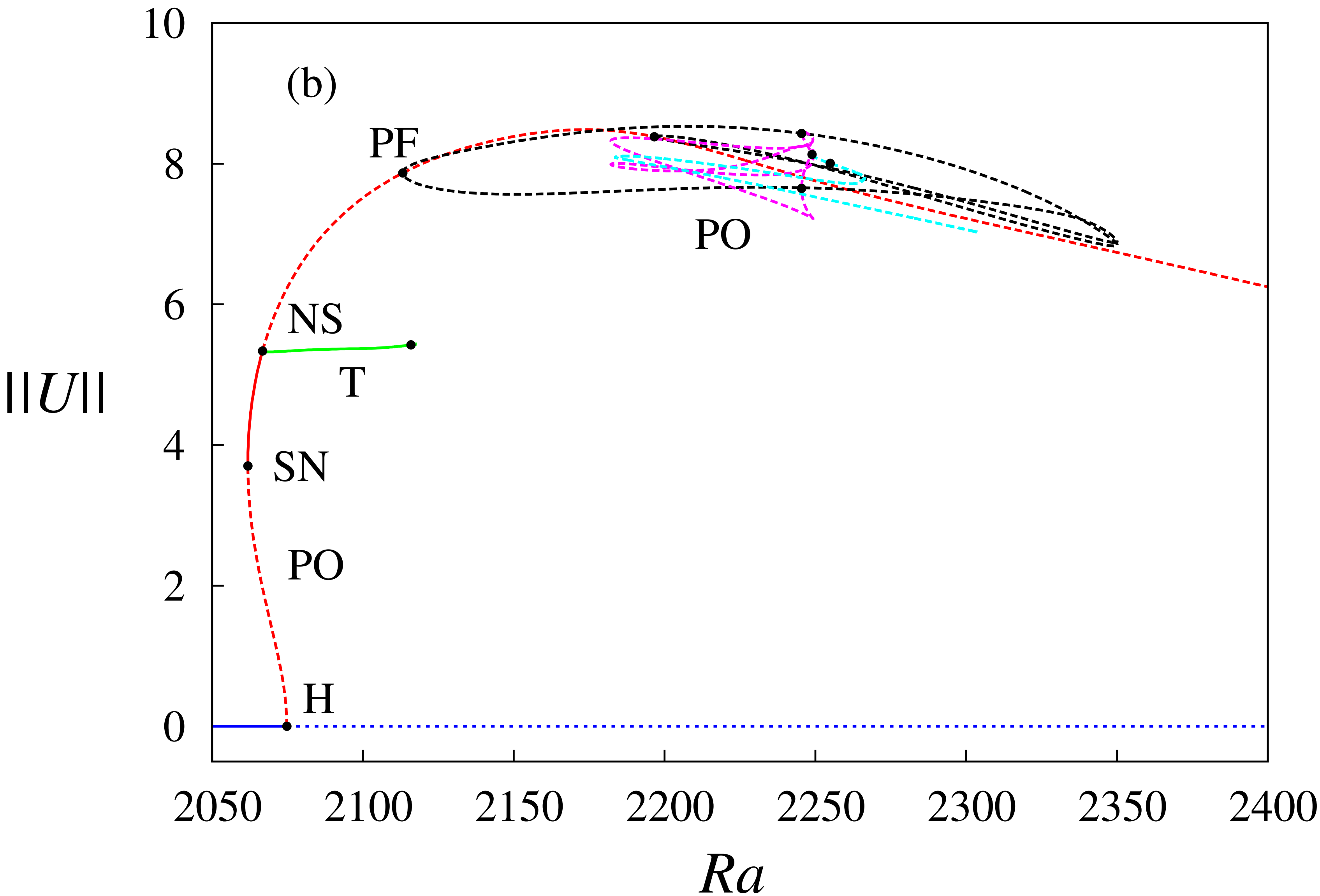
$$\frac{1}{\Delta t} B \left(\gamma_0 u^{n+1} - \sum_{i=0}^{k-1} \alpha_i u^{n-i} \right) = \sum_{i=0}^{k-1} \beta_i N(u^{n-i}) + Lu^{n+1}.$$

The initial points are obtained by a fully implicit BDF method.

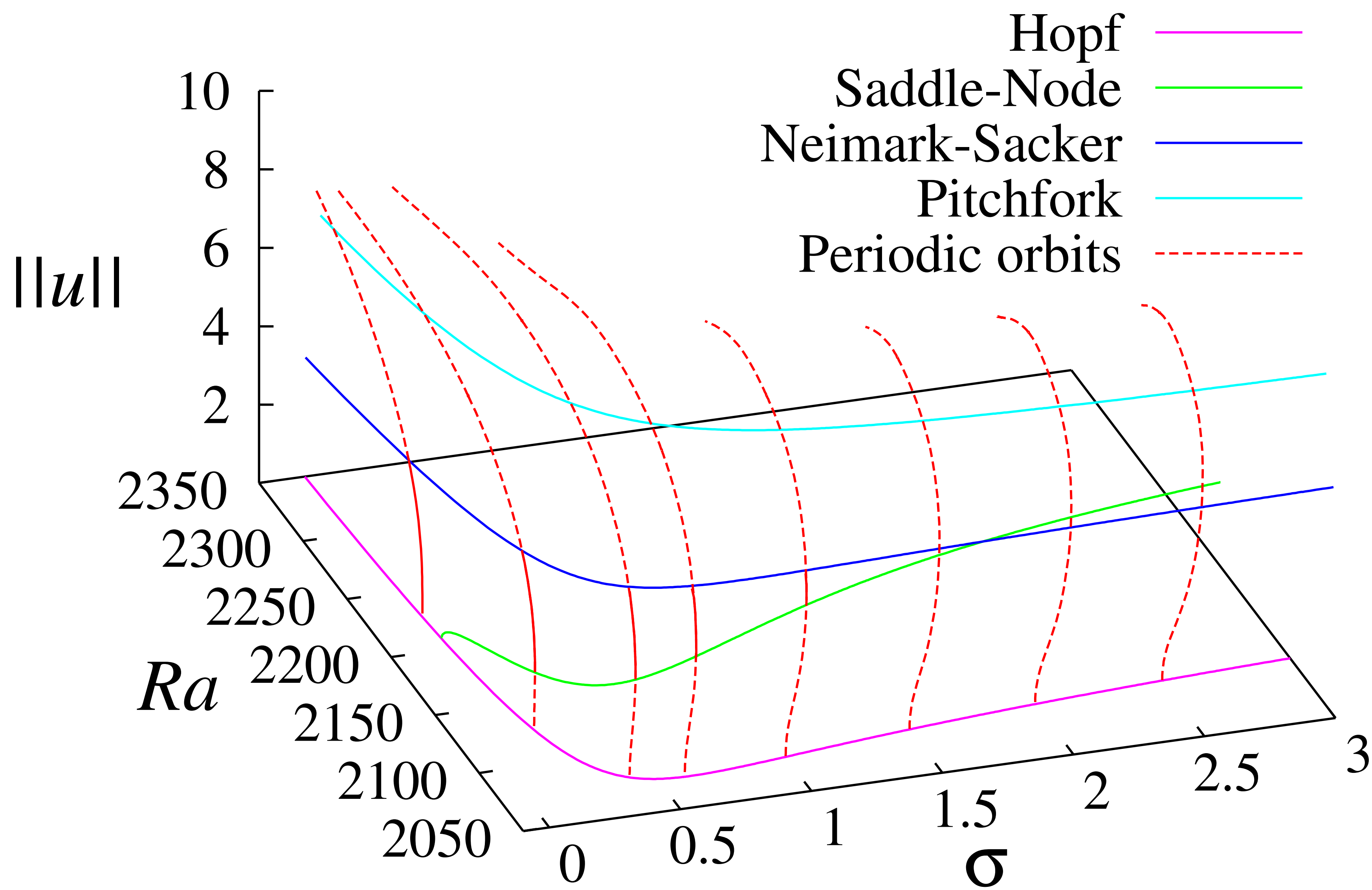
Some results for $\sigma = 0.6$



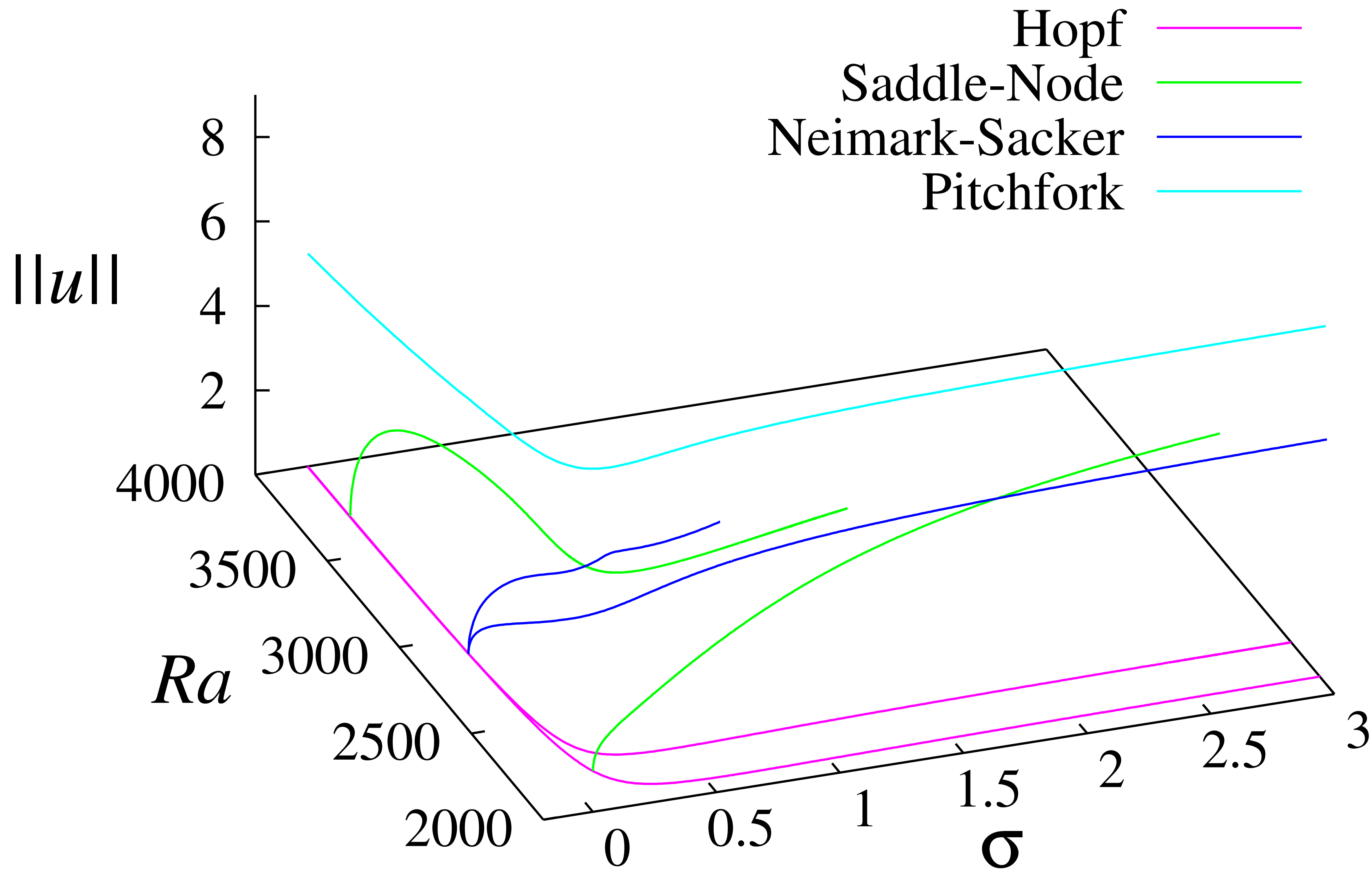
Some results for $\sigma = 0.6$



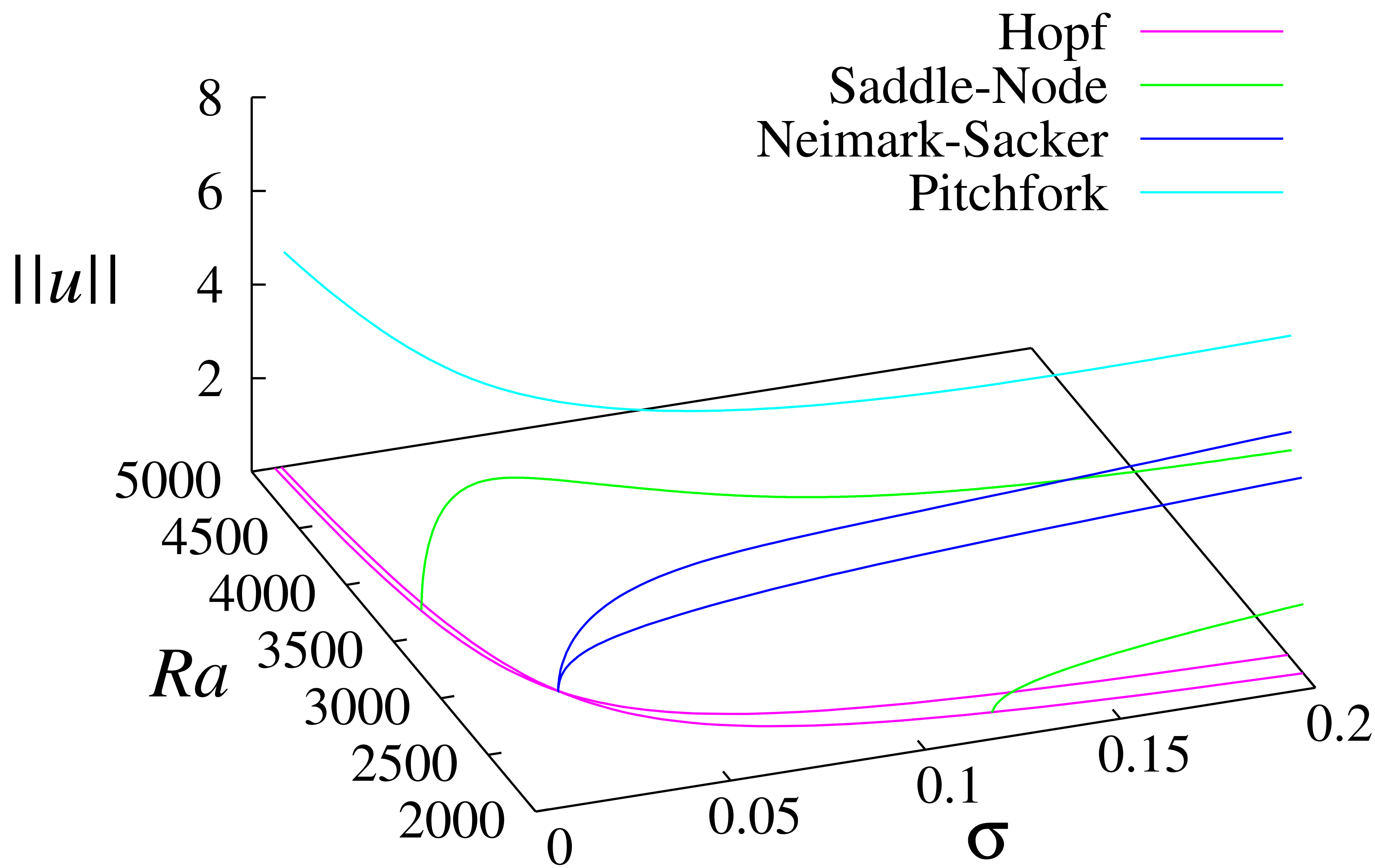
Curves of bifurcations



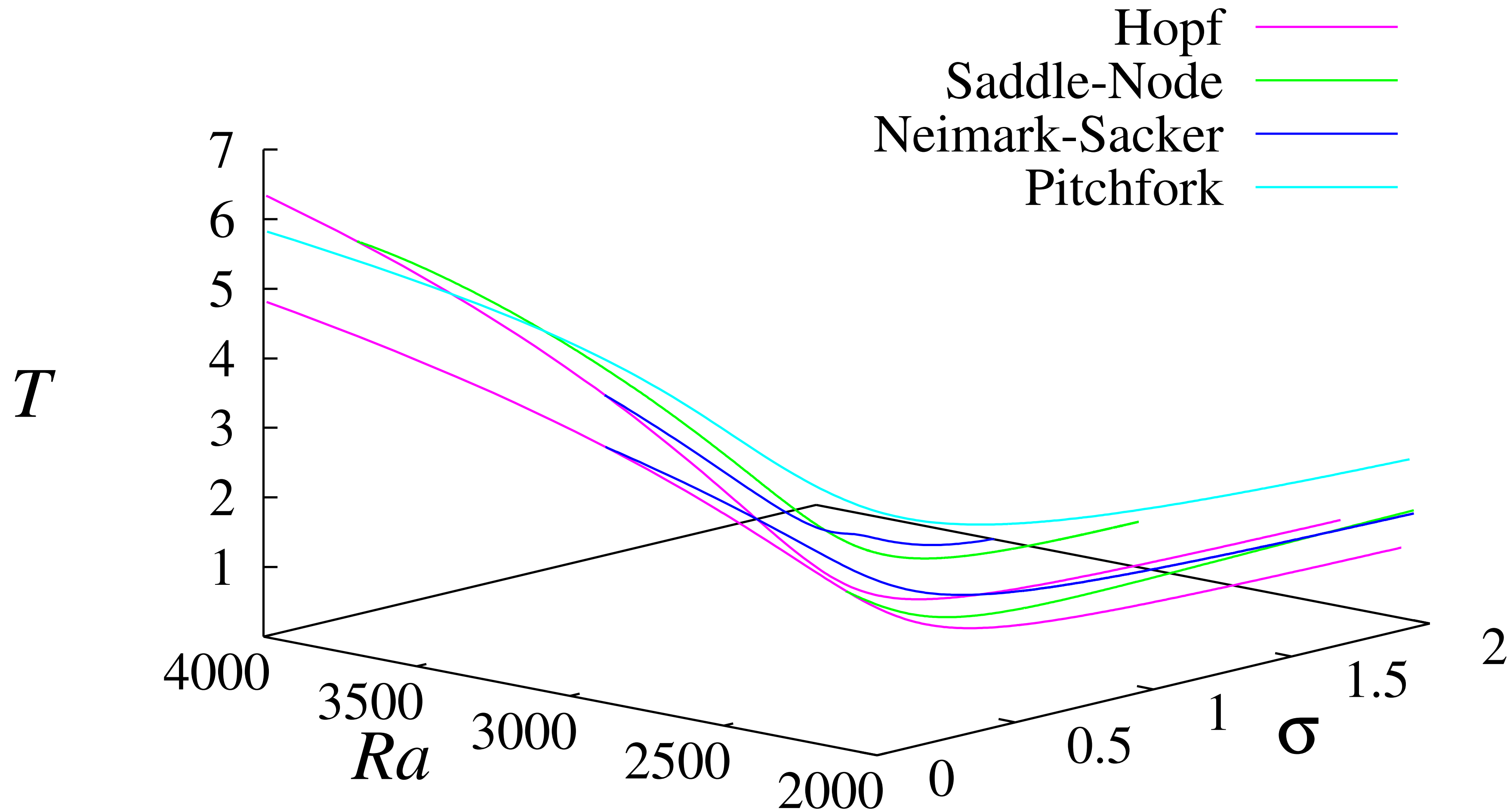
Curves of bifurcations



Curves of bifurcations

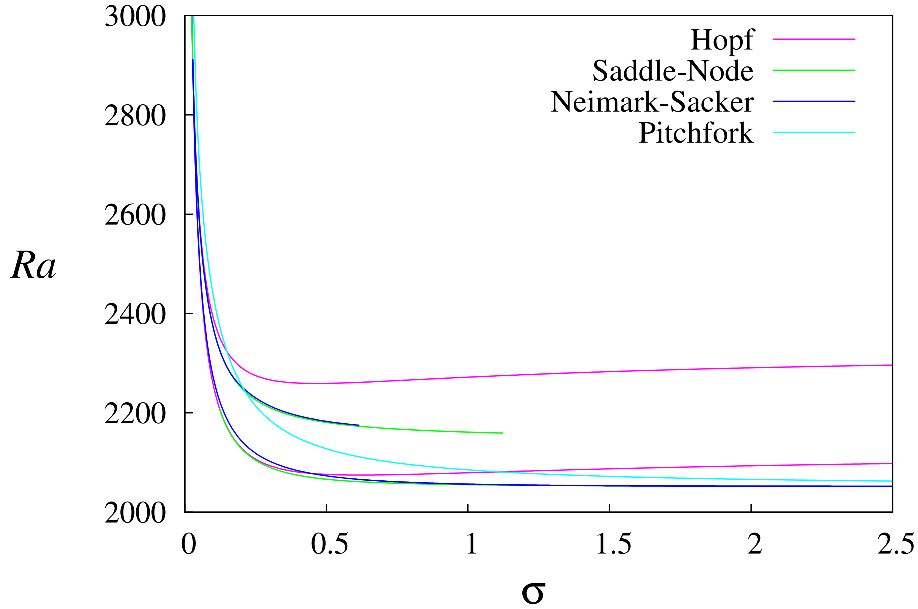


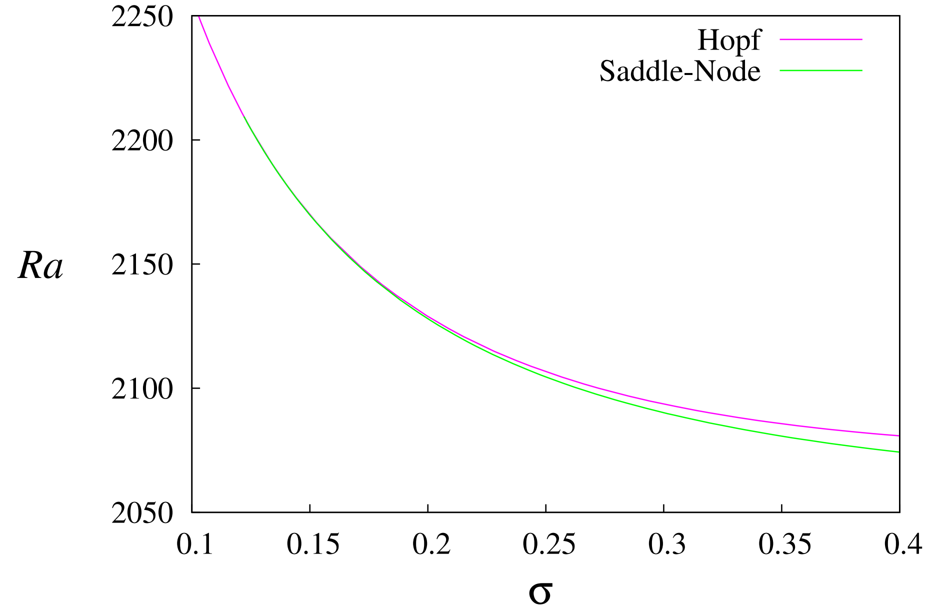
Period

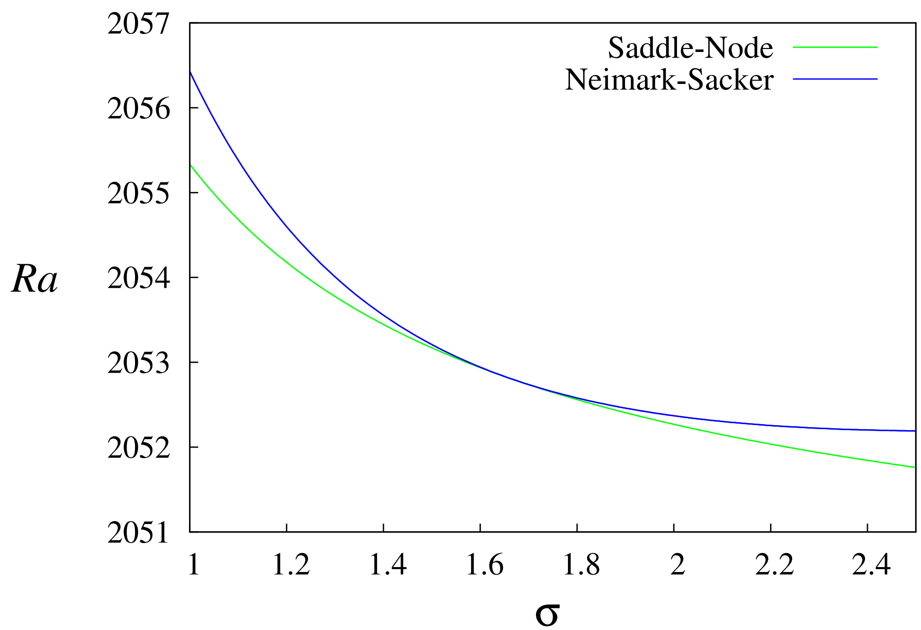


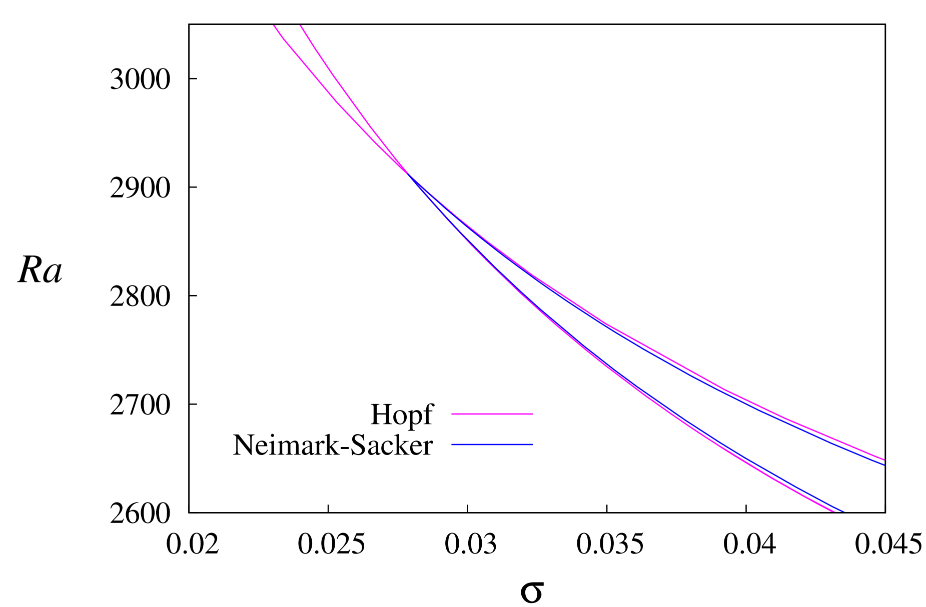
Hopf
Saddle-Node
Neimark-Sacker
Pitchfork
7
6
5
4
3
2
1
T
4000
3500
3000
2500
2000
Ra
0
0.5
1
1.5
2
σ

Codimension-two points

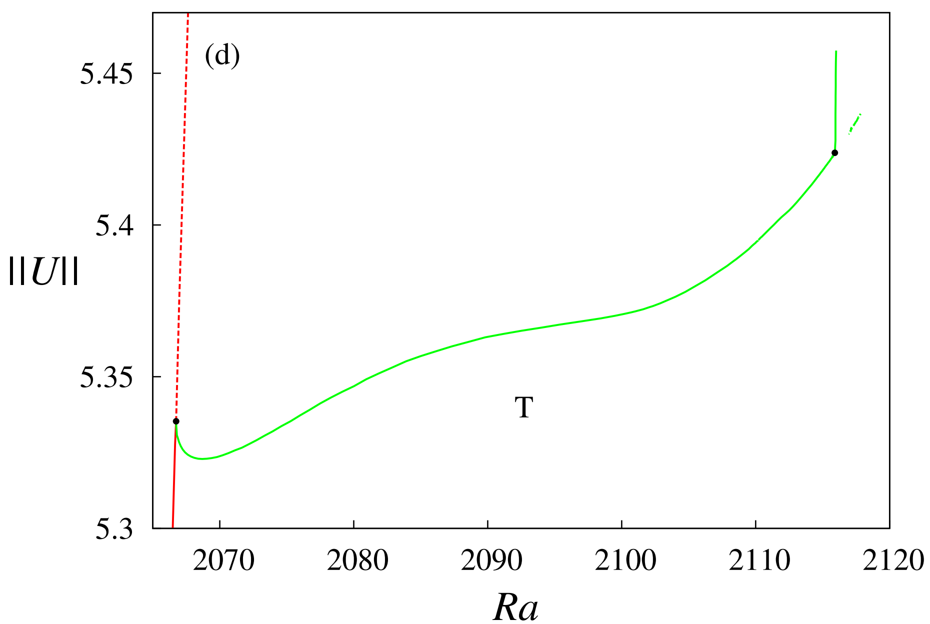




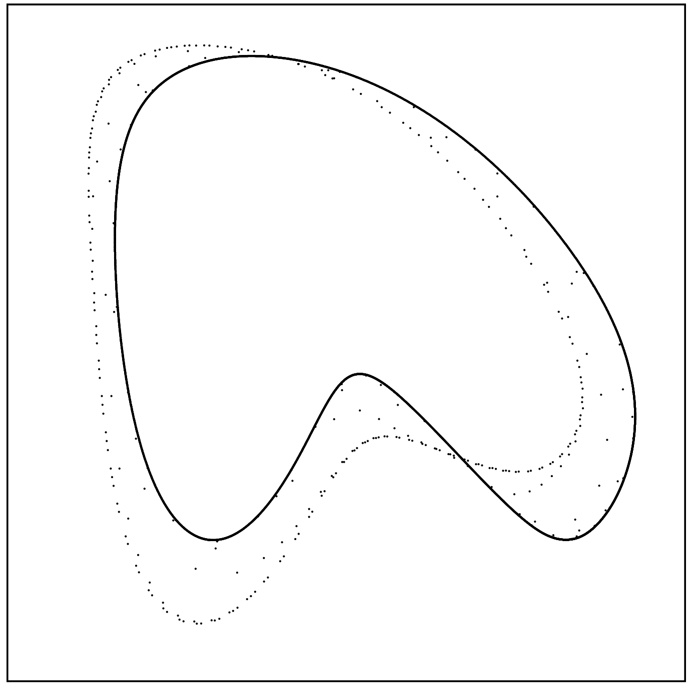




Invariant tori for $\sigma = 0.6$

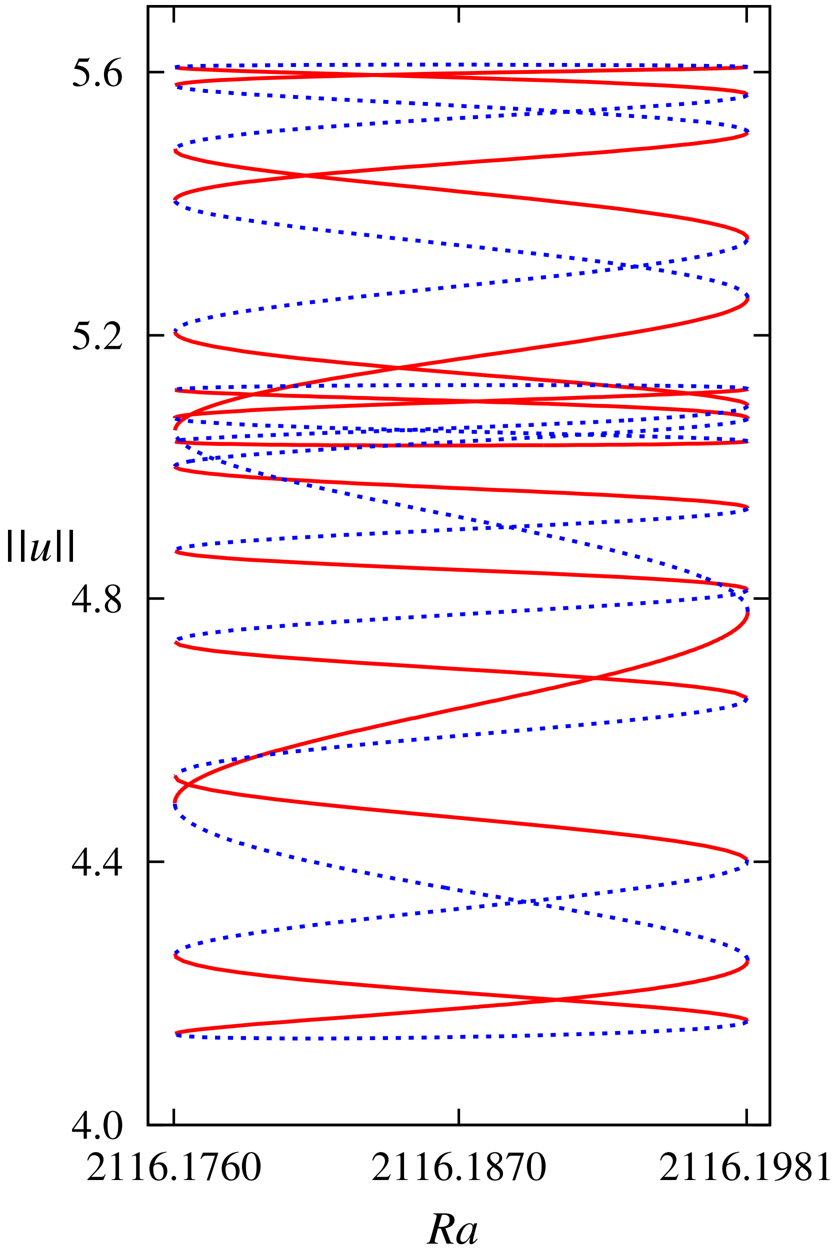


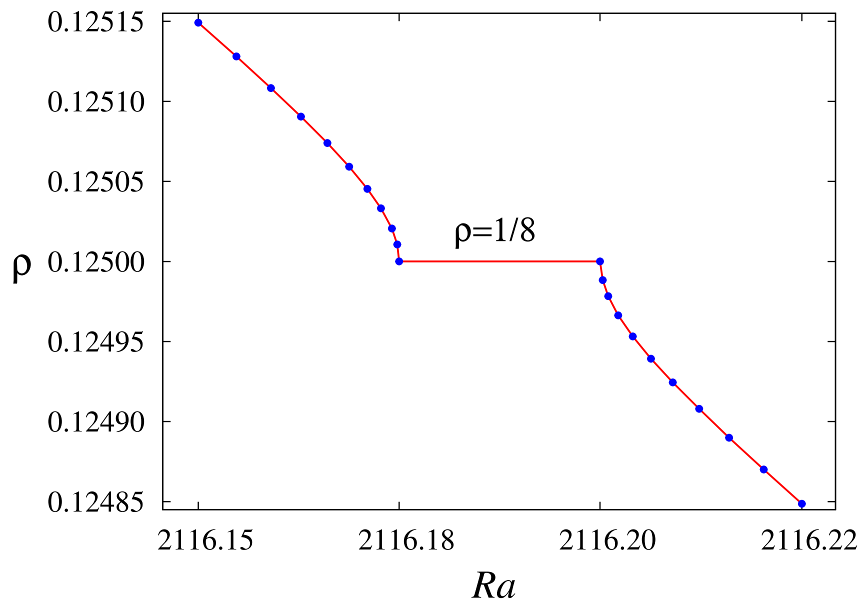
Ra=2117.4954

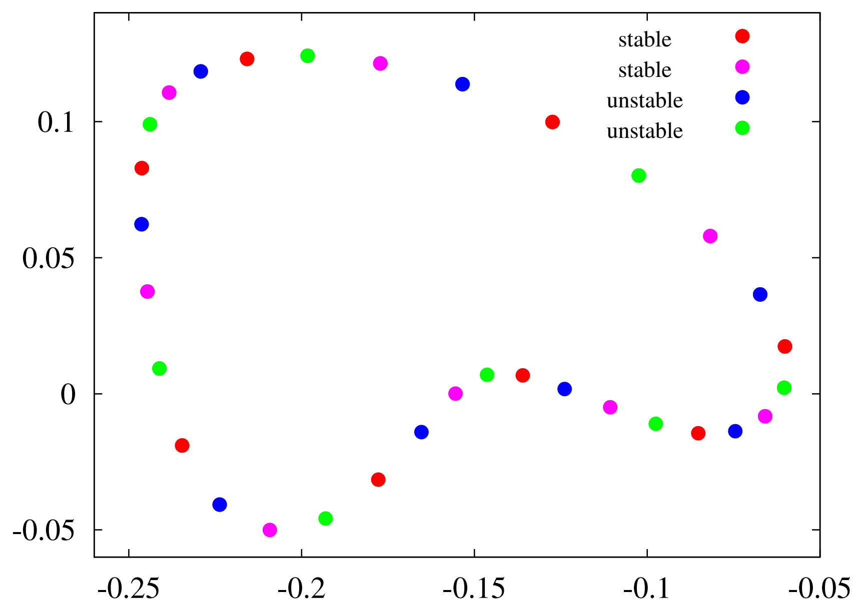


- Beginning of the branch: $Ra = 2066.74$
- 1/7–resonance interval $2102.79 < Ra < 2102.80$
- Pitchfork bifurcation $Ra \approx 2115.92$
- 1/8–resonance interval $2116.18 \leq Ra \leq 2116.20$.
- First period doubling $Ra \approx 2118.40$
- Second period doubling $Ra \approx 2118.55$
- Breakdown of the torus $Ra \approx 2118.60$

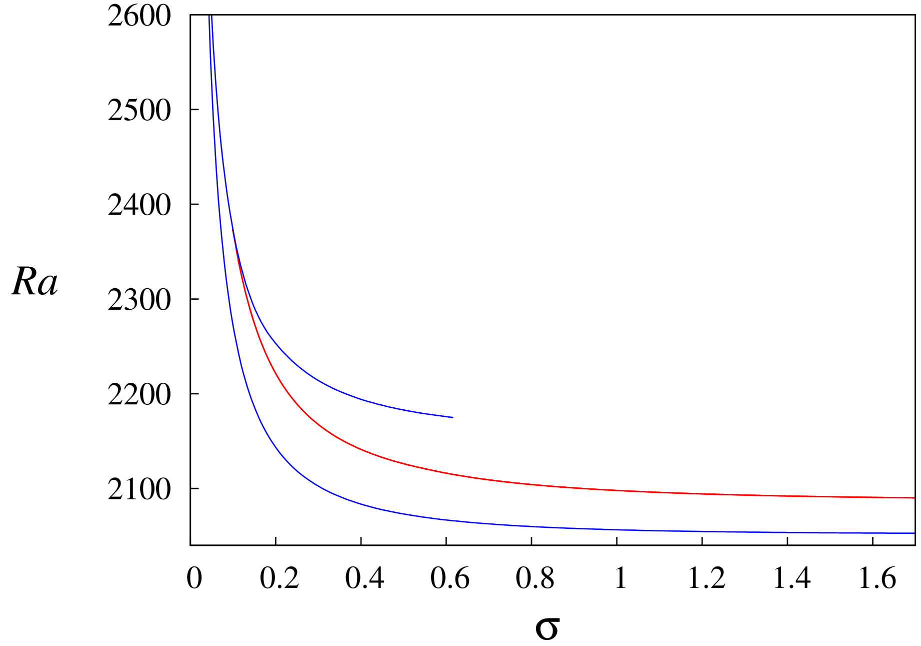
The Arnold's tongue of $\rho = 1/8$ $(\sigma = 0.6)$

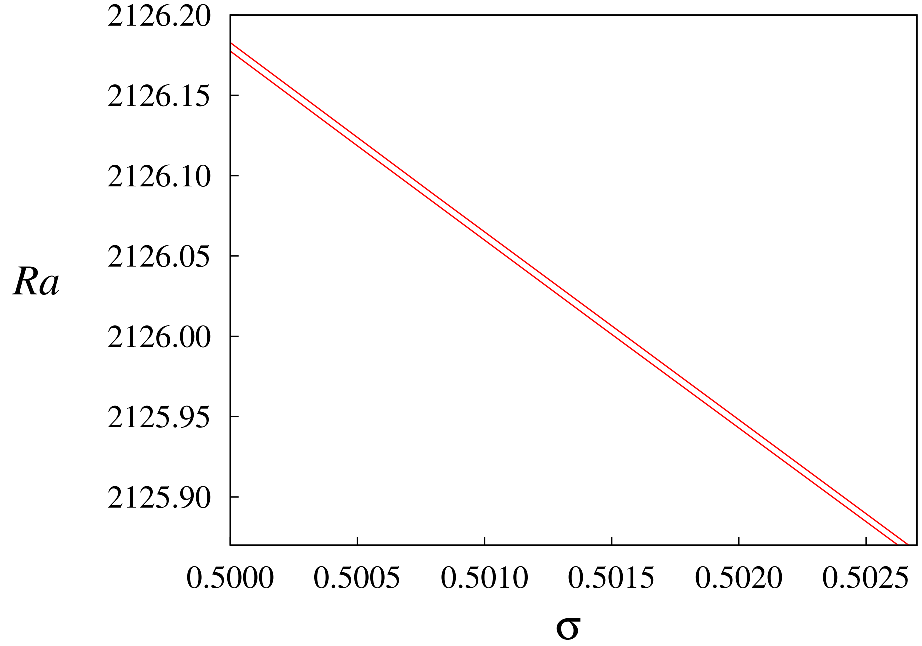


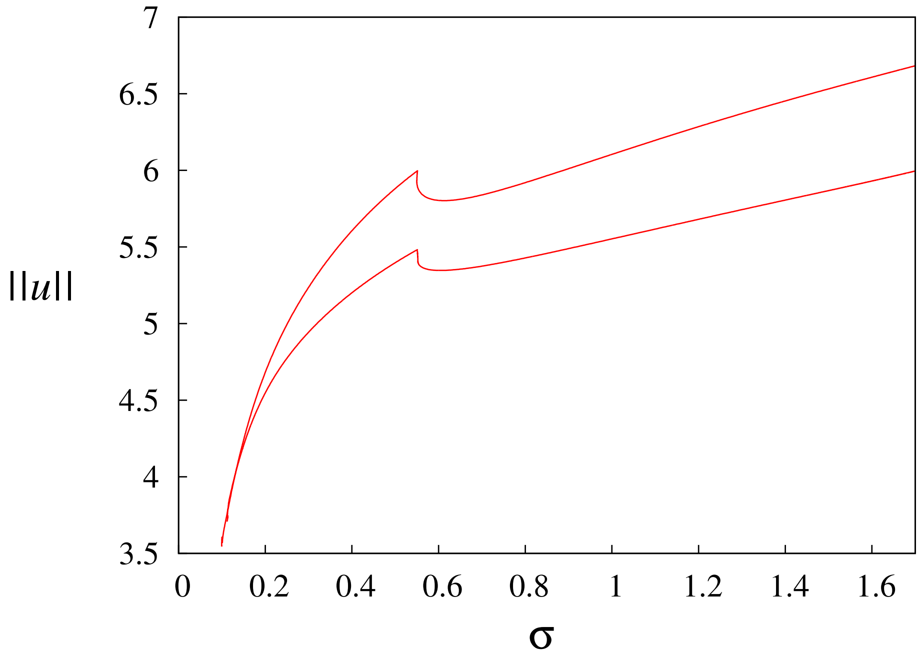


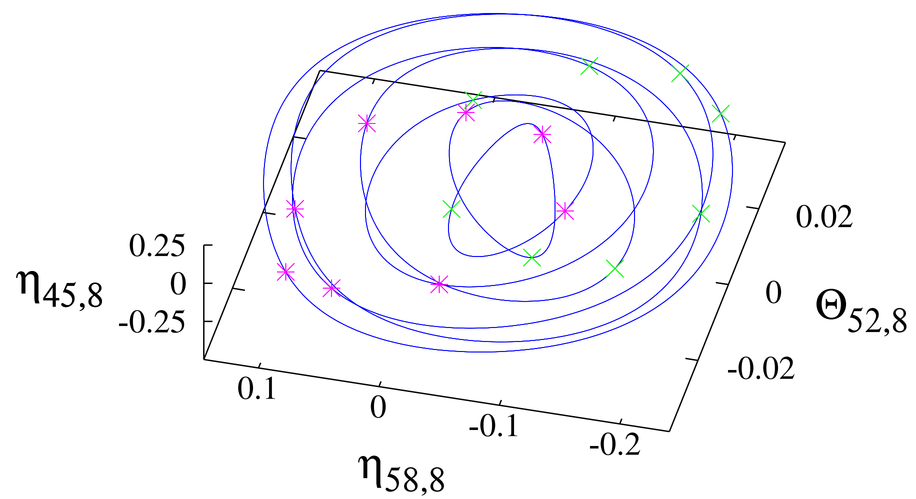


Computation of the limits of the $1/8$ tongue

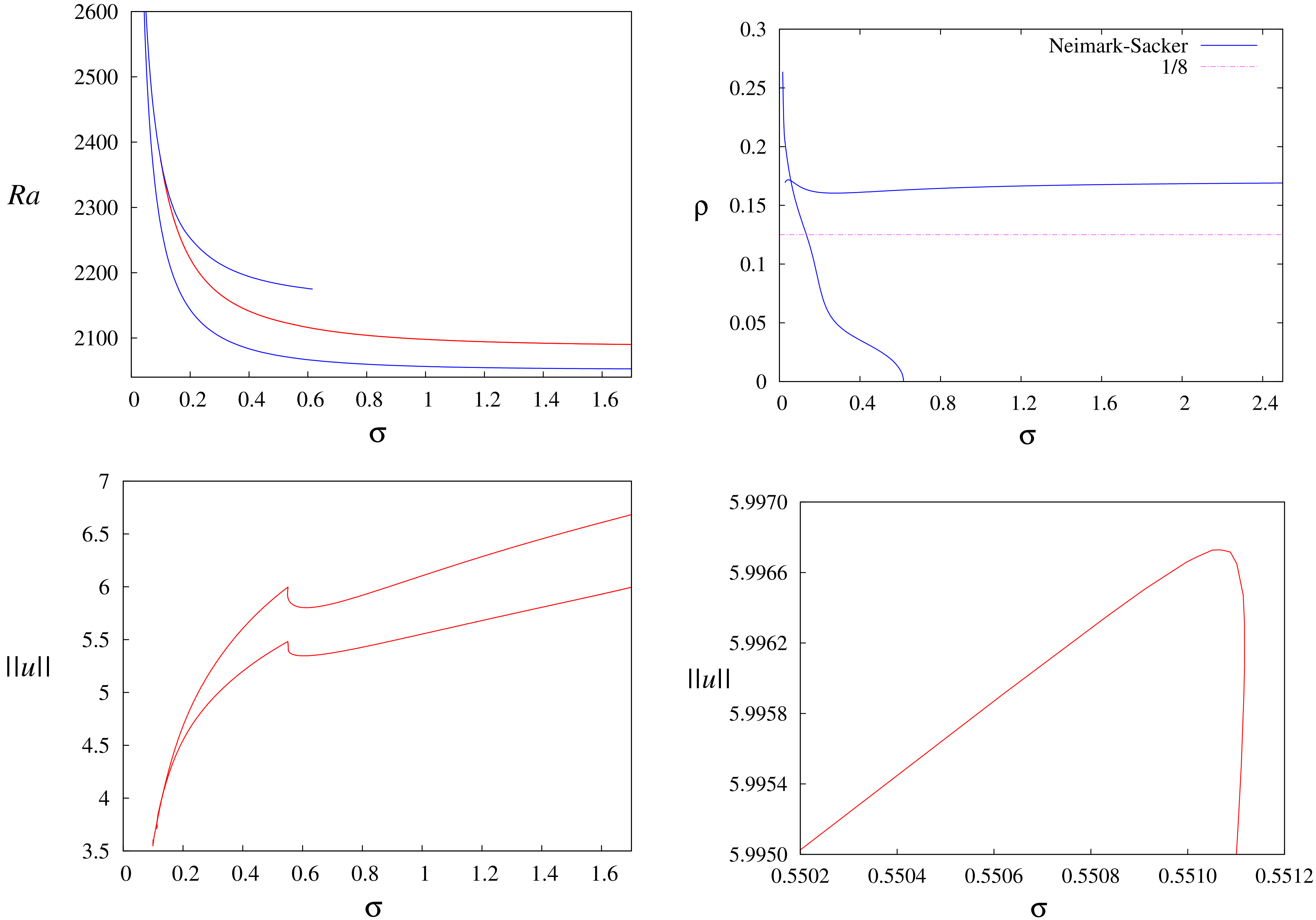




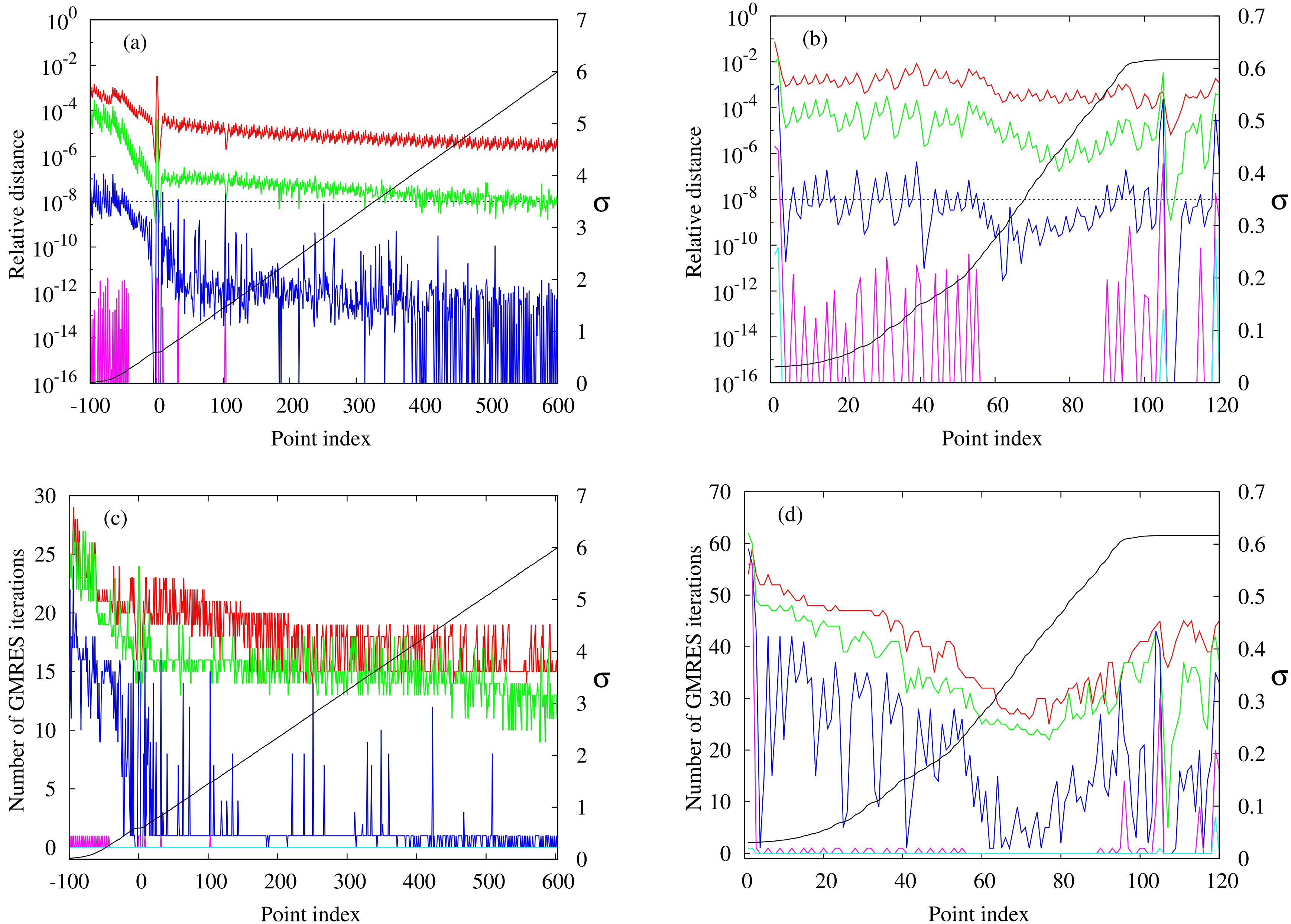




Computation of the limits of the $1/8$ tongue



Performance



Relative distance between Newton iterates and number of GMRES iterations for the pitchfork and one of the Neimark-Sacker curves.

Reference

- Net M., Sánchez J. *Continuation of bifurcations of periodic orbits of dissipative PDEs*, SIAM J. Appl. Dyn. Syst. **14**, 678–698, 2015.