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Spectral and Transport Properties of One-Dimensional Disordered Conductors (Notas de Curso)

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Prólogo

La investigación de las propiedades físicas de medios desordenados (aleatorios) representa uno de los problemas de mayor actualidad en la Física moderna. Uno de los más importantes descubrimientos del siglo XX en este campo es el relacionado con el fenómeno de localización dinámica de ondas clásicas y partículas cuánticas. Resulta que su movimiento en un campo potencial aleatorio frecuentemente ocurre en un espacio confinado (localizándose) incluso cuando la energía cinética de la partícula (onda) es mucho mayor que su energía potencial aleatoria. En otras palabras, el fenómeno de localización dinámica es profundamente cuántico y tiene lugar en condiciones de la así llamada reflexión sobre barrera. La localización de electrones en conductores desordenados fue predicha teóricamente por Anderson en 1958, por lo que se conoce ahora como localización de Anderson o fuerte. La localización de Anderson se manifiesta en las propiedades de conducción (transporte) por el hecho que el metal deja de conducir la corriente eléctrica, es decir se vuelve prácticamente un aislante.

La segunda característica notable de un sistema de electrones que no interactúan entre sí consiste en que sus propiedades de conducción (transporte) están determinadas por la dimensión del espacio. Es bien sabido que en los conductores tridimensionales comunes la resistencia residual (es decir la resistencia a la temperatura del cero absoluto) está determinada por la longitud de recorrido libre de los electrones. Esta longitud se debe a su dispersión elástica con defectos de diferente naturaleza en la muestra. Si no hubiese dispersión, es decir si la muestra fuera un medio ordenado ideal, entonces la resistencia sería igual a cero. Resulta que en conductores unidimensionales aún en la ausencia de cualquier dispersión la resistencia residual y su cantidad inversa, la conductancia, siempre son finitas. Más aún, ellas no dependen de los parámetros del conductor, sino que son una combinación de constantes universales: la carga del electrón e, la constante de Planck $\hbar$, el número π. Es así como surgió una constante universal nueva: el cuanto de la conductancia, $e^2/\pi\hbar$. El transporte electrónico bajo las condiciones en que la dispersión de los electrones puede despreciarse se llama balístico.

Actualmente está rigurosamente establecido que en los conductores unidimensionales desordenados se realiza ya sea el régimen balístico o el de localización, dependiendo de la relación entre la longitud del conductor y la así llamada longitud de localización. Esta longitud de localización está determinada por las características estadísticas del potencial aleatorio dispersor (su amplitud y rango de variación), así como las propiedades del sistema electrónico (energía de Fermi). Si la longitud del conductor es mucho menor que la longitud de localización, se lleva a cabo el régimen de transporte balístico. En el caso contrario tiene lugar la localización fuerte de los estados electrónicos y la conductancia resulta ser una cantidad exponencialmente pequeña del cociente de la longitud de la muestra entre la longitud de localización.

Como conclusión de esta breve descripción del transporte electrónico en conductores desordenados unidimensionales queda solamente agregar lo siguiente. De hecho, cualquier sólido es un medio desordenado. Incluso los cristales lo son gracias a la presencia de diferentes tipos de defectos en la red cristalina (vacancias, impurezas, dislocaciones, interfaces microcristalinas, etc.). Además, resulta que es imposible, aún teóricamente, crear una red cristalina unidimensional estrictamente ordenada. Tal creación es absolutamente inestable y se transforma en un sistema aleatoriamente desordenado bajo la acción de cualquier perturbación, cuan pequeña que esta sea.

En el curso que se impartirá se exponen los fundamentos de la teoría de transporte electrónico en conductores unidimensionales desordenados. La solución de cualquier problema comienza con el análisis del espectro uni-partícula del sistema investigado. Por esta razón, el presente curso consta de dos partes. En la primera, se calcula una característica uni-partícula importante de los electrones: su función de Green promediada. Con su ayuda se introducen tales cantidades espectrales básicas como la longitud de recorrido libre total (saliente) de los electrones, las longitudes electrónicas de dispersión hacia adelante y hacia atrás. En la segunda parte, se expone sistemáticamente, precisamente, la teoría del transporte electrónico. Los resultados obtenidos describen los regímenes tanto balístico como de localización, así como la transición entre ellos. Se llega a una importante conclusión según la cual el conductor unidimensional desordenado representa un ejemplo típico de los sistemas mesoscópicos.

Una característica importante del presente curso consiste en que no sólo se discute el problema físico, sino que en él también se enseñan métodos de la Física Teórica, los cuales pueden ser utilizados en otros campos de investigación. Así, por ejemplo, en la primera parte del curso se expone detalladamente el método ya tradicional para resolver la ecuación de Dyson, se introducen los conceptos de operador de auto-energía. El procedimiento de promediación se lleva a cabo con ayuda de la técnica elegante, recientemente desarrollada por Maradudin y colaboradores. En la segunda parte del curso se presenta el modelo de oscilaciones de escala doble, los métodos de promedicación de fases rápidas y el cálculo de los correladores (funciones de correlación) para campos aleatorios. Aquí se formulan los elementos de la teoría general de dispersión de ondas en medios aleatorios, se introducen los conceptos de reflectancia y transmitancia, así como sus momentos. En la segunda parte del curso también se presenta la fórmula de Furutsu-Novikov y aprendemos a utilizarla en la deducción de las ecuaciones necesarias para los momentos de reflectancia. Finalmente, se muestra el método de las funciones de distribución el cual es un instrumento potente para resolver ecuaciones diferenciales y de diferencias, y a su vez es ampliamente aplicado en problemas de muchas partículas en la Física moderna de sistemas desordenados.

Las presentes notas de curso están dirigidas a los estudiantes de los últimos cursos de la carrera en Física, a los estudiantes de posgrado y a todos los físicos interesados en problemas de la Física moderna.

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Chapter 1

Green's Function of 1D Disordered Conductor

1.1 General concepts

In these lectures we discuss distinctive spectral and conducting properties of one-dimensional (1D) electron systems. Such systems possess two significant features. From the physical point, the effect of strong (Anderson) localization of electron states due to electron scattering from impurities is particularly pronounced in 1D conductors. From the mathematical point, the theory of the 1D localization has been built completely.

One of the most important spectral characteristics of a quantum system is *single-particle Green's function*. The physical meaning of single-particle Green's function is the probability for a free particle placed at a given point of space to reach another given point in specified time. As good texts for getting familiar with Green's functions we can recommend a very accessible and illustrative book [1] and more rigorous and advanced book [2]. It is remarkable that once the Coulomb repulsion between electrons can be neglected, the conductivity or conductance of an electron system can be expressed in terms of certain products of two single-particle Green's functions. Therefore it seems plausible to approach the conducting properties of 1D conductors by considering first single-particle Green's function.

To start with, let us consider the following Schrödinger equation:

$$(\varepsilon - \mathcal{H})\,\Psi = 0, \tag{1.1}$$

where $\mathcal{H}$ is the Hamiltonian, ε is the energy and Ψ is the wave function of an electron.

The Green function (or the Green operator) $\mathcal{G}$ of the Schrödinger equation (1.1) is a solution of the corresponding inhomogeneous equation with the unit-operator on the right-hand side (r.h.s.):

$$(\varepsilon - \mathcal{H})\,\mathcal{G}(\varepsilon) = \mathbf{1}. \tag{1.2}$$

Obviously, a formal solution of this equation is the following inverse operator:

$$\mathcal{G}(\varepsilon) = (\varepsilon - \mathcal{H})^{-1}. \tag{1.3}$$

One can consider this equality as a definition of the Green function $\mathcal{G}(\varepsilon)$.

Using Eq. (1.3) we can rewrite the Green function $\mathcal{G}(\varepsilon)$ in the representation of the eigenfunctions Ψ_a of the Hamiltonian $\mathcal{H}$. By definition, in this representation the Hamiltonian $\mathcal{H}$ is a diagonal matrix, i.e.

$$\mathcal{H}\Psi_a = \varepsilon_a \Psi_a. \tag{1.4}$$

The Green function $\mathcal{G}(\varepsilon)$ turns out to be a diagonal matrix as well:

$$\mathcal{G}_a(\varepsilon) = \frac{1}{\varepsilon - \varepsilon_a}. \tag{1.5}$$

Here the symbol "a" labels matrix elements and denotes a complete set of quantum numbers (both continuous and discrete) of the problem, ε_a is an eigenvalue of the Hamiltonian $\mathcal{H}$ corresponding to a quantum number a.

In the coordinate space defined by the position vector $\vec{r}$ the unit-operator $\mathbf{1}$ is described by the Dirac delta-function $\delta(\vec{r} - \vec{r'})$ and *the equation for the Green function* $\mathcal{G}(\varepsilon)$ is rewritten as:

$$(\varepsilon - \mathcal{H}_{\vec{r}})\, \mathcal{G}(\vec{r}, \vec{r'}; \varepsilon) = \delta(\vec{r} - \vec{r'}). \tag{1.6}$$

Since the Hamiltonian $\mathcal{H}$ of any physically meaningful quantum system is an Hermitian operator, its eigenfunctions $\Psi_a(\vec{r})$ form a *complete* set of orthogonal and normalized (i.e. *orthonormal*) functions that can be chosen as a basis in the Hilbert space of complex-valued functions of $\vec{r}$ [3]. Mathematically, the orthonormalization and completeness of the set of functions $\Psi_a(\vec{r})$ are defined as

$$\int d\vec{r}\, \Psi^*_{a'}(\vec{r}) \Psi_a(\vec{r}) = \delta_{aa'}, \qquad \sum_a \Psi_a(\vec{r}) \Psi^*_a(\vec{r'}) = \delta(\vec{r} - \vec{r'}), \tag{1.7}$$

respectively. The l.h.s of the first equation is nothing else but the scalar product (dot product) of the wave functions $\Psi_{a'}(\vec{r})$ and $\Psi_a(\vec{r})$. The asterisk "$*$"stands for complex conjugation. In general, the symbol $\delta_{aa'}$ on the r.h.s. of the first equation implies the product of Dirac delta-functions and Kronecker delta-symbols. Dirac delta-functions are related to continuous eigenvalues of the spectrum $\{a\}$, while Kronecker delta-symbols are associated with its discrete part. The second equation in Eq. (1.7) (the completeness condition) follows from the first one and represents an expansion of the Dirac delta-function $\delta(\vec{r} - \vec{r'})$ in the basis $\Psi_a(\vec{r})$. The symbol of the sum over a implies an integration over continuous eigenvalues and summation over discrete eigennumbers.

According to the equation (1.6) and conditions (1.7), the Green function $\mathcal{G}(\vec{r}, \vec{r'}; \varepsilon)$ can be presented as *the Hilbert–Schmidt series*:

$$\mathcal{G}(\vec{r}, \vec{r'}; \varepsilon) = \sum_a \frac{\Psi_a(\vec{r}) \Psi^*_a(\vec{r'})}{\varepsilon - \varepsilon_a}. \tag{1.8}$$

This series is the expansion of the Green function $\mathcal{G}(\vec{r}, \vec{r'}; \varepsilon)$ in the complete set of the orthonormal wave functions $\Psi_a(\vec{r})$.

From the formula (1.8), as well as from the representation (1.5), we see that *the Green function $\mathcal{G}(\varepsilon)$ has singularities at the values of the external energy ε equal to any of the eigenvalues ε_a*, i.e. at $\varepsilon = \varepsilon_a$. In the representation (1.5) these singularities are simple poles, whereas in the coordinate representation (1.8) they can be weaker due to summation or integration over the complete set $\{a\}$. Thus, as it was mentioned above, *if*

one calculates the Green function of a quantum system and reveals its singularities, these singularities provide complete information about the spectral properties of the system.

We note that the quantity ε is an external parameter in the Green function problem. It has the meaning of energy and therefore is real in all physical applications. Owing to the hermicity of the Hamiltonian $\mathcal{H}$, the eigenvalues ε_a are also real. Hence the above mentioned poles $\varepsilon = \varepsilon_a$ lie exactly on the path of the summation (integration) in the Hilbert–Schmidt series (1.8). Since the summand (integrand) diverges at the poles, a problem of correct taking the poles $\varepsilon = \varepsilon_a$ into account arises. Usually this problem is solved by going around poles along infinitely small contours. Surprisingly, the Green function depends on a direction of going around the poles. As a consequence of this dependence, three different Green's functions were introduced: *retarded* $\mathcal{G}_+(\varepsilon)$*, advanced* $\mathcal{G}_-(\varepsilon)$*, and causal* $\mathcal{G}_c(\varepsilon)$.

The retarded $\mathcal{G}_+(\varepsilon)$ *and advanced* $\mathcal{G}_-(\varepsilon)$ *Green functions* satisfy the following equations:

$$(\varepsilon \pm i0 - \mathcal{H}_{\vec{r}})\,\mathcal{G}_\pm(\vec{r},\vec{r'};\varepsilon) = \delta(\vec{r}-\vec{r'}), \tag{1.9}$$

whence the Hilbert–Schmidt series for them take the form:

$$\mathcal{G}_\pm(\vec{r},\vec{r'};\varepsilon) = \sum_a \frac{\Psi_a(\vec{r})\Psi_a^*(\vec{r'})}{\varepsilon - \varepsilon_a \pm i0}. \tag{1.10}$$

Hereinafter the symbols "$\pm i0$" stand for an arbitrarily small, tending to zero, positive ($+i0$) or negative ($-i0$) imaginary quantity. In other words, the symbol "$+i0$" implies approaching the real axis of the complex energy plane from above, while the symbol "$-i0$" requires to approach the real axis from below. We point out that the energy parameter ε is considered real in Eqs. (1.9) and (1.10).

So, we see that *the energy* ε *acquires a positive infinitely small imaginary shift for the retarded Green function* $\mathcal{G}_+(\varepsilon)$ *and a negative infinitely small imaginary shift for the advanced Green function* $\mathcal{G}_-(\varepsilon)$.

The retarded $\mathcal{G}_+(\varepsilon)$ and advanced $\mathcal{G}_-(\varepsilon)$ Green functions are related as

$$\mathcal{G}_\pm^*(\vec{r},\vec{r'};\varepsilon) = \mathcal{G}_\mp(\vec{r'},\vec{r};\varepsilon). \tag{1.11}$$

The causal Green function $\mathcal{G}_c(\varepsilon)$ is defined as the difference between the advanced $\mathcal{G}_-(\varepsilon)$ and retarded $\mathcal{G}_+(\varepsilon)$ Green functions:

$$\mathcal{G}_c(\varepsilon) = \mathcal{G}_-(\varepsilon) - \mathcal{G}_+(\varepsilon). \tag{1.12}$$

Let us substitute the equalities (1.10) into the definition (1.12) for the causal Green function $\mathcal{G}_c(\varepsilon)$ and use the following Dirac identity:

$$2\pi i\delta(\varepsilon - \varepsilon_a) \equiv \frac{1}{\varepsilon - \varepsilon_a - i0} - \frac{1}{\varepsilon - \varepsilon_a + i0}. \tag{1.13}$$

As a result, we obtain the Hilbert–Schmidt series for the causal Green function $\mathcal{G}_c(\varepsilon)$:

$$\mathcal{G}_c(\vec{r},\vec{r'};\varepsilon) = 2\pi i \sum_a \delta(\varepsilon - \varepsilon_a)\Psi_a(\vec{r})\Psi_a^*(\vec{r'}). \tag{1.14}$$

In accordance with Eq. (1.14) the causal Green function $\mathcal{G}_c(\varepsilon)$ has an evident property:

$$\mathcal{G}_c^*(\vec{r},\vec{r'};\varepsilon) = -\mathcal{G}_c(\vec{r'},\vec{r};\varepsilon) \tag{1.15}$$

Since all three Green's functions introduced above can be uniquely expressed via each other, it suffices to analyze only one of them to find out properties of all the three functions. Following a commonly accepted tradition, we will discuss below the retarded Green function $\mathcal{G}_+(\varepsilon)$ for our 1D disordered problem. For simplicity, we will omit the subscript "+" at the retarded Green function. In addition to this, we will leave out the positive infinitely small imaginary energy shift $+i0$ every time when it does not lead to misunderstanding.

1.2 Problem Statement

We consider a 1D wire (straight line) of infinite length. The x axis is directed along the wire. We assume that there always exist conduction electrons of fixed concentration in the wire. As far as the spectral and conducting properties are of our interest, we will deal with the conduction electrons only. For simplicity, we choose the free electron model for the conduction electrons. Within this model the dispersion law of electrons is assumed to be quadratic and isotropic,

$$\varepsilon = \hbar^2 k^2/2m. \tag{1.16}$$

Here m is the effective mass and $\vec{k}$ is the wave vector of electrons.

Our goal is to study the Green function of a disordered (imperfect) 1D wire. As a physically plausible model for disorder we introduce the random potential $V(x)$ that specifies the potential energy of electrons due to disorder at any point x of the wire. We assume that $V(x)$ is a continuous, statistically homogeneous and isotropic random process with zero mean value. These assumptions lead to the following correlation properties of the potential $V(x)$:

$$\langle V(x)\rangle = 0, \qquad \langle V^2(x)\rangle = V_0^2, \qquad \langle V(x)V(x')\rangle = V_0^2\mathcal{W}(|x-x'|). \tag{1.17}$$

The angular brackets $\langle\ldots\rangle$ stand for statistical averaging over the ensemble of realizations of the random function $V(x)$. Here V_0 is the root-mean-square (r.m.s.) value and V_0^2 is the dispersion of the potential $V(x)$. The function $\mathcal{W}(|x|)$ is the binary coefficient of correlation characterized by the unit amplitude $\mathcal{W}(0) = 1$ and by the scale R_c of monotonous decrease. The scale R_c is called the correlation radius. The dispersion V_0^2 of the potential $V(x)$ does not depend on the coordinate x, while the correlation coefficient $\mathcal{W}(|x-x'|)$ depends only on the distance between points x and x'. These two facts are direct consequences of the statistical homogeneity and isotropy of the random process (i.e., the homogeneity and isotropy of our problem on average). The random potential $V(x)$ is the only source of electron scattering in our problem.

An important and relatively well studied mechanism of electron scattering in low dimensional solid-state devices is scattering from non-magnetic static impurities. The concentration of impurities is usually sufficiently low so that the average distance between neighboring scattering centers can be quite large. Therefore successive collisions of electron with impurities are statistically independent and the correlation between them is absent. In this case the random potential $V(x)$ is regarded as the Gaussian "white noise",

i.e. a process with extremely small correlation radius R_c. As a result, for electron-impurity scattering the binary coefficient of correlation $\mathcal{W}(|x|)$ is replaced by the Dirac delta-function:

$$\mathcal{W}(|x-x'|) = R_c\delta(x-x'). \tag{1.18}$$

Such random processes are known as random delta-correlated processes.

According to all the above assumptions, the 1D Hamiltonian $\mathcal{H}_{\vec{r}}$ takes the form:

$$\mathcal{H}_{\vec{r}} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x). \tag{1.19}$$

Therefore the equation for the retarded Green function $\mathcal{G}(x,x';\varepsilon)$ is written as

$$\left(\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \varepsilon + i0\right)\mathcal{G}(x,x';\varepsilon) - V(x)\mathcal{G}(x,x';\varepsilon) = \delta(x-x'). \tag{1.20}$$

To get rid of the constant factor $\hbar^2/2m$ in subsequent calculations it is convenient to introduce the renormalized Green function $\mathcal{G}^{new}(x,x';k)$ and renormalized random potential $V^{new}(x)$:

$$\mathcal{G}^{new}(x,x';k) = (\hbar^2/2m)\mathcal{G}^{old}(x,x';\varepsilon); \tag{1.21}$$

$$V^{new}(x) = (2m/\hbar^2)V^{old}(x), \qquad V_0^{new} = (2m/\hbar^2)V_0^{old}. \tag{1.22}$$

Note that the new random potential $V^{new}(x)$ has the dimension of squared wave number. It should be emphasized that $V^{new}(x)$ possesses the same statistical properties (1.17) with the same binary coefficient of correlation $\mathcal{W}(|x|)$ as the "old" potential $V(x)$.

We will discuss below the new retarded Green function omitting the superscript "*new*" for simplicity. So, according to definitions (1.21) and (1.22) and Eq. (1.16) we obtain the following *equation for the retarded Green function of a 1D disordered conductor*:

$$\left(\frac{d^2}{dx^2} + k^2 + i0\right)\mathcal{G}(x,x';k) - V(x)\mathcal{G}(x,x';k) = \delta(x-x'). \tag{1.23}$$

The equation (1.23) should be complemented by *boundary conditions*. As the boundary conditions we adopt reasonable requirements of finiteness of the Green function at the infinitely remote points $x = \pm\infty$:

$$|\mathcal{G}(x=\pm\infty,x')| < \infty. \tag{1.24}$$

Such boundary conditions are often called *the radiative conditions*. We remind that a problem with boundary (not initial and not periodic) conditions is called *boundary-value problem or problem with open ends*.

In accordance with Eq. (1.23), the Green function satisfies the corresponding homogeneous equation at $x \neq x'$, i.e. Eq. (1.23) with zero in place of the delta-function on the r.h.s. At the point $x = x'$ the Green function is continuous but its first derivative with respect to x has a unit jump. Let us write down *the conditions for the continuity of the Green function and for the unit jump of the derivative*:

$$\mathcal{G}(x = x' + 0, x') \;-\; \mathcal{G}(x = x' - 0, x') = 0,$$

$$\mathcal{G}'(x = x' + 0, x') \;-\; \mathcal{G}'(x = x' - 0, x') = 1. \tag{1.25}$$

Hereinafter a prime at a function stands for the derivative of this function with respect to x. The symbols "± 0" denote an arbitrarily small, tending to zero, positive ($+0$) or negative (-0) quantity. In other words, the symbol "-0" ("$+0$") implies taking the left-hand (right-hand) limit. The second of Eqs. (1.25) is derived by integrating Eq. (1.23) over x over the interval $(x' - 0, x' + 0)$. We emphasize that the Green function of *any* 2nd order differential equation with a unit factor standing at the second derivative satisfies the conditions (1.25).

Thus, we always have two ways to solve the Green function problem. The first way is to solve the inhomogeneous equation (1.23) with two boundary conditions (1.24). The second one is to solve the corresponding homogeneous equation with four conditions: two boundary conditions (1.24) and two conditions (1.25) at the internal point $x = x'$.

In most practically important cases one is interested in the Green function averaged over the ensemble of realizations of the random potential $V(x)$ rather than in the exact Green function itself. Moreover, often one is unable to solve the problem exactly because of an unknown explicit form of the potential $V(x)$. What *is* known are the statistical properties (1.17) of the random process $V(x)$. Therefore, our goal here is to find the *average* Green function $\langle \mathcal{G}(x, x'; k) \rangle$ of the boundary-value problem (1.23), (1.24), (1.25).

1.3 Unperturbed Problem (Perfect Wire, No Disorder)

First of all, let us analyze the corresponding unperturbed boundary-value problem, i.e. the problem with zero random potential ($V(x)$=0). In this case *the retarded Green function* $\mathcal{G}_0(x, x'; k)$ *satisfies the equation*

$$\left(\frac{d^2}{dx^2} + k^2 + i0 \right) \mathcal{G}_0(x, x'; k) = \delta(x - x') \tag{1.26}$$

and the radiative boundary conditions (1.24) at $x = \pm\infty$.

Certainly, *at the point* $x = x'$ *the unperturbed Green function* $\mathcal{G}_0(x, x'; k)$ *satisfies the conditions* (1.25) *for the continuity of the Green function and for the unit jump of its derivative.*

Now we will write down the solution of the homogeneous boundary-value problem (1.26), (1.24), (1.25). We postpone the rigorous derivation of the solution until next subsections. The retarded Green function $\mathcal{G}_0(x, x'; k)$ of a 1D homogeneous (perfect) conductor has the following form:

$$\begin{aligned} \mathcal{G}_0(x - x'; k) &= \frac{1}{2ik} \left\{ \exp[ik(x' - x)]\Theta(x' - x) + \exp[ik(x - x')]\Theta(x - x') \right\} = \\ &= \frac{\exp(ik|x - x'|)}{2ik}, \qquad k \to k + i0. \end{aligned} \tag{1.27}$$

Here $\Theta(x)$ is the Heaviside unit-step Θ-function. One should keep in mind that the symbol k should be regarded as $k + i0$.

We see that the unperturbed Green function $\mathcal{G}_0(x-x';k)$ depends only on the distance between points x and x'. This fact is caused by the *homogeneity and isotropy of the unperturbed problem.*

1.3.1 Traditional construction of solution

There exists a "traditional" method for finding the Green function of a 1D boundary-value problem (i.e. a problem with open ends). This method is based on solving the corresponding homogeneous equation with four conditions: two boundary conditions and two conditions (1.25) for the continuity of the Green function and for the unit jump of its derivative at the point $x = x'$.

First of all, we will explain briefly the essence of the method and then apply it to our unperturbed problem (1.26), (1.24), (1.25).

1. *General construction of solution:*

Our objective is to obtain the Green function of a 2nd order homogeneous differential equation with a unit factor standing at the second derivative and boundary conditions formulated at the ends of some interval.

Let $\psi_\pm(x)$ be two linearly independent solutions of the homogeneous equation, chosen so that $\psi_-(x)$ satisfies the boundary condition at the left open end and $\psi_+(x)$ at the right open end of the interval.

Then the Green function is built as follows:

$$\begin{aligned} \mathcal{G}(x < x') &= \frac{1}{W}\psi_-(x)\psi_+(x'), \\ \mathcal{G}(x > x') &= \frac{1}{W}\psi_+(x)\psi_-(x'). \end{aligned} \tag{1.28}$$

Here W is the Wronskian of the functions $\psi_\pm(x)$ defined as

$$W = \psi_-(x')\psi'_+(x') - \psi_+(x')\psi'_-(x'). \tag{1.29}$$

We point out that the Wronskian of any Hermitian problem does not depend on the coordinate x'.

It is easily verifiable that the expression (1.28) is the Green function of the problem. Indeed, it satisfies the initial differential equation and boundary conditions due to the proper chosen functions $\psi_\pm(x)$. Evidently, at the point $x = x'$ the function (1.28) satisfies the continuity and unit-jump conditions (1.25). We note that the unit amplitude of the jump has been achieved by introducing the normalization factor in the form of the Wronskian (1.29).

Thus, the problem of deriving an explicit formula for the Green function reduces to seeking two linearly independent solutions $\psi_\pm(x)$ of the homogeneous equation. Note that these solutions are defined up to a constant factor which cancels out in Eq. (1.28) owing to the Wronskian.

2. *Solutions $\psi_\pm(x)$ of the unperturbed problem* (1.26), (1.24):

Obviously, two linearly independent solutions $\psi_\pm(x)$ of the homogeneous equation (1.26) can be written in the form of plane waves:

$$\psi_{\pm}(x) = \exp(\pm ikx), \qquad k \to k + i0. \tag{1.30}$$

The plane wave $\psi_+(x) = \exp(ikx)$ propagates to the right while the plane wave $\psi_-(x) = \exp(-ikx)$ propagates to the left. It is easy to verify that the function $\psi_-(x) = \exp(-ikx)$ satisfies the boundary condition (1.24) at the left open end $x = -\infty$ and the function $\psi_+(x) = \exp(ikx)$ satisfies the boundary condition (1.24) at the right open end $x = +\infty$ of the wire. To do this we should take into account that the symbol k stands for $k + i0$.

By the definition (1.29), the Wronskian of the "left" and "right" plane waves (1.30) is

$$W = 2ik. \tag{1.31}$$

After the substitution of the solutions (1.30) and their Wronskian (1.31) into the general expression (1.28) we arrive at the explicit expression (1.27) for the Green function $\mathcal{G}_0(x - x'; k)$.

1.3.2 Fourier representation

Since our problem is defined within a continuous infinite interval ($-\infty < x < \infty$), we can seek the unperturbed Green function $\mathcal{G}_0(x - x'; k)$ in the form of the double Fourier integral,

$$\mathcal{G}_0(x - x') = \int_{-\infty}^{\infty} \frac{dk_x\, dk'_x}{(2\pi)^2} \exp(ik_x x) G_0(k_x, k'_x) \exp(-ik'_x x'), \tag{1.32}$$

The Fourier transform $G_0(k_x, k'_x)$ is defined as

$$G_0(k_x, k'_x) = \int_{-\infty}^{\infty} dx\, dx' \exp(-ik_x x) \mathcal{G}_0(x - x') \exp(ik'_x x'). \tag{1.33}$$

So, to find the Green function $\mathcal{G}_0(x - x'; k)$ we need to calculate the Fourier transform $G_0(k_x, k'_x)$ from the initial equation (1.26). To this end, let us apply the integral Fourier operator

$$\int_{-\infty}^{\infty} dx\, dx' \exp[-i(k_x x - k'_x x')] \tag{1.34}$$

to the both sides of this equation. Then we should perform a double integration by parts for the first term on the l.h.s. of the equation. At the same time, we should use the following representations for the Dirac delta-functions:

$$\begin{aligned} \delta(x - x') &= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \exp[\pm ik_x(x - x')], \\ \delta(k_x - k'_x) &= \int_{-\infty}^{\infty} \frac{dx}{2\pi} \exp[\pm i(k_x - k'_x)x]. \end{aligned} \tag{1.35}$$

It is interesting that *the first equality* in the representations (1.35) *is the condition of completeness* and *the second one is the condition of orthonormalization* for the basis $\Psi_{k_x}(x) = (2\pi)^{-1/2} \exp(ik_x x)$. Indeed, for our 1D homogeneous problem (1.26), (1.24) defined within the continuous infinite interval ($-\infty < x < \infty$) the complete set of eigenvalues $\{a\}$ consists of the single continuous wave number k_x and the eigenfunctions $\Psi_a(\vec{r})$

are $\Psi_{k_x}(x)$. Therefore the equalities (1.35) coincide with the corresponding conditions from Eq. (1.7) when rewritten explicitly for our case.

As a result of the above calculations, we obtain the following.

Owing to the homogeneity of the unperturbed problem, the Green matrix $\hat{\mathcal{G}}_0(k)$ turns out to be diagonal in the k_x-representation:

$$G_0(k_x, k'_x) = G_0(k_x)\, 2\pi\delta(k_x - k'_x), \tag{1.36}$$

$$G_0(k_x) = \frac{1}{k^2 - k_x^2 + i0}. \tag{1.37}$$

Therefore, the inverse (1.32) and direct (1.33) Fourier transforms assume simpler forms:

$$\begin{aligned} \mathcal{G}_0(x - x') &= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, G_0(k_x) \exp[ik_x(x - x')], \\ G_0(k_x) &= \int_{-\infty}^{\infty} dx\, \mathcal{G}_0(x) \exp(-ik_x x). \end{aligned} \tag{1.38}$$

Thus, the Fourier integral expansion for the Green function $\mathcal{G}_0(x - x'; k)$ of the unperturbed problem (1.26) and (1.24) reads as:

$$\mathcal{G}_0(x - x'; k) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \frac{\exp[ik_x(x - x')]}{k^2 - k_x^2 + i0}. \tag{1.39}$$

We emphasize that this Fourier representation is nothing else but the Hilbert-Schmidt series (1.10) for the 1D retarded Green's function defined within the continuous infinite interval $(-\infty < x < \infty)$.

Evidently, the Fourier transform (1.37) and consequently the representation (1.39) can be obtained by direct applying the Fourier transformations (1.33) or (1.38) to the explicit formula (1.27). On the other hand, we can come to the explicit expression (1.27) by evaluating the integral in the Fourier representation (1.39). Let us do it now.

We should take the Fourier integral over k_x in equation (1.39) according to the residue theorem from the complex-variable theory. Details of the calculation can be found in any textbook on the complex-variable theory, e.g. [4].

As a first step, we need to find the singularities of the integrand. The integrand has only *two simple poles* which are determined by equating the denominator to zero:

$$k^2 - k_x^2 + i0 = 0 \qquad \rightarrow \qquad k_x = \pm(k + i0). \tag{1.40}$$

It is very important that our external parameter $k^2 + i0$ has a positive imaginary part. Owing to this fact the poles do not lie on the integration path which is the real axis of the complex plane k_x. The first pole $k_x = k + i0$ is in the upper half-plane and the second one $k_x = -k - i0$ lies in the lower half-plane.

Let us calculate the residues of the integrand at the points $k_x = \pm(k + i0)$. To do that, we should expand the integrand in the Laurent series in the vicinity of these poles:

$$\frac{1}{2\pi}\, \frac{\exp[ik_x(x - x')]}{k^2 - k_x^2 + i0} \approx \mp \frac{1}{4\pi k}\, \frac{\exp[\pm ik(x - x')]}{k_x \mp (k + i0)},$$

$$\text{at} \qquad |k_x \mp (k + i0)| \ll 1. \tag{1.41}$$

By definition, the residues of the integrand are the factors at $[k_x \mp (k + i0)]^{-1}$ in the expansion (1.41):

$$\text{Res}\{integrand(k_x = \pm(k + i0))\} = \mp\frac{\exp[\pm ik(x - x')]}{4\pi k}. \tag{1.42}$$

Now we are in a position to evaluate the Fourier integral (1.39).

For definiteness, suppose that the difference $x - x'$ is positive ($x - x' > 0$). In this case the integrand is an analytical function of the integration variable k_x within the upper half-plane of the complex plane k_x. Therefore, we can make the integration path closed by adding to its ends $k_x = \pm\infty$ an infinitely far semi-circle R_+ in the upper half-plane. In this way we obtain a new closed contour C_+ that consists of the initial integration path $-\infty < k_x < \infty$ and the infinitely far semi-circle R_+ in the upper half-plane. We see that the first pole $k_x = k + i0$ only lies within the closed contour C_+. So, according to the residue theorem, the integral along the closed contour C_+ is equal to $2\pi i$ multiplied by the integrand residue at the point $k_x = k + i0$:

$$\int_{C_+} \frac{dk_x}{2\pi} \frac{\exp[ik_x(x - x')]}{k^2 - k_x^2 + i0} = \frac{\exp[ik(x - x')]}{2ik} \qquad \text{when} \qquad x - x' > 0. \tag{1.43}$$

We next take into account that the integral along the infinitely far semi-circle R_+ vanishes because the exponent $\exp[ik_x(x-x')]$ for $x-x' > 0$ goes to zero at any infinitely far point in the upper half-plane of the complex plane k_x. Therefore the Fourier integral (1.39) is equal to the integral (1.43) along the closed contour C_+. As a result we get

$$\begin{aligned} \mathcal{G}_0(x - x'; k) &= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{\exp[ik_x(x - x')]}{k^2 - k_x^2 + i0} = \\ &= \frac{\exp[ik(x - x')]}{2ik} \qquad \text{when} \qquad x - x' > 0. \end{aligned} \tag{1.44}$$

If the difference $x-x'$ is negative ($x-x' < 0$), then the integral (1.39) can be taken in a similar manner except for the integrand being now an analytical function of k_x in the *lower* (not upper) half-plane of the complex plane k_x. Therefore we can modify the integration path by closing it with the infinitely far semi-circle R_- in the lower half-plane. Hence, we get a new closed contour C_- that consists of the initial integration path $-\infty < k_x < \infty$ and the infinitely far semi-circle R_- in the lower half-plane. Now only the second pole $k_x = -k - i0$ lies within the closed contour C_-. So, in line with the residue theorem, the integral along this closed contour C_- is equal to $-2\pi i$ multiplied by the integrand residue at the point $k_x = -k - i0$:

$$\int_{C_-} \frac{dk_x}{2\pi} \frac{\exp[ik_x(x - x')]}{k^2 - k_x^2 + i0} = \frac{\exp[-ik(x - x')]}{2ik} \qquad \text{for} \qquad x - x' < 0. \tag{1.45}$$

Note that we have used the factor $-2\pi i$ instead of $2\pi i$ because we went along the contour C_- clockwise but not counter-clockwise as we did for positive $x-x'$. The integral along the

infinitely far semi-circle R_- vanishes because the exponent $\exp[ik_x(x-x')]$ for $x-x'<0$ is equal to zero at any infinitely far point in the lower half-plane of the complex plane k_x. Therefore the initial integral (1.39) is equal to the integral (1.45) along the closed contour C_-. As a result we get

$$\begin{aligned} \mathcal{G}_0(x-x';k) &= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{\exp[ik_x(x-x')]}{k^2-k_x^2+i0} = \\ &= \frac{\exp[-ik(x-x')]}{2ik} \qquad \text{for} \qquad x-x'<0. \end{aligned} \tag{1.46}$$

From a straightforward comparison one can make sure that the expressions (1.44) and (1.46) are identical with the equation (1.27) obtained earlier for the unperturbed Green function.

1.4 Problem with Disorder

Now let us return to the derivation of the averaged Green function $\langle \mathcal{G}(x,x';k)\rangle$ of the 1D disordered conductor. At the beginning we can state that the average Green function $\langle \mathcal{G}(x,x';k)\rangle$ depends only on the difference $x-x'$ of coordinates owing to the statistical homogeneity of the problem on average, i.e., owing to the statistical homogeneity (1.17) of the scattering potential $V(x)$. So, we can write

$$\langle \mathcal{G}(x,x';k)\rangle = \overline{\mathcal{G}}(x-x';k). \tag{1.47}$$

In these Lecture Notes we will obtain the average Green function via deriving and solving *the Dyson equation*.

First of all, we need to derive the starting *Green formula* to relate the perturbed Green function $\mathcal{G}(x,x';k)$ to the unperturbed Green function $\mathcal{G}_0(x,x';k)$. Note that the Green formula is nothing else but the closed integral equation for $\mathcal{G}(x,x';k)$. To derive it we will use the initial differential equation (1.23) with the radiative boundary conditions (1.24).

We now describe briefly how to obtain the Green formula. First, we write down the equation (1.26) for $\mathcal{G}_0(x_1,x;k)$ and the equation (1.23) for $\mathcal{G}(x_1,x';k)$. Then we multiply the former from the left through by $\mathcal{G}(x_1,x';k)$, while the latter by $\mathcal{G}_0(x_1,x;k)$. We next subtract the latter from the former cancelling out identical terms. The result is integrated over x_1 over the interval $(-\infty,\infty)$. Terms that include derivatives are cancelled after integration by parts. Finally, we apply the symmetry property $\mathcal{G}_0(x',x;k)=\mathcal{G}_0(x,x';k)$. In this way we arrive at the required Green formula which has the usual form:

$$\mathcal{G}(x,x';k) = \mathcal{G}_0(x-x';k) + \int_{-\infty}^{\infty} dx_1 \mathcal{G}_0(x-x_1;k)V(x_1)\mathcal{G}(x_1,x';k). \tag{1.48}$$

Next, from the Green formula (1.48) for the exact (i.e., random) Green function $\mathcal{G}(x,x';k)$ we should derive an equation for the average Green function $\langle \mathcal{G}(x,x';k)\rangle$. To this end, we can try some different ways. For example, we can average directly both sides of the Green formula (1.48), in which case the correlator $\langle V(x_1)\mathcal{G}(x_1,x';k)\rangle$ appears which can unlikely be decoupled in a plausible way.

A second way is to apply a diagrammatic technique similar to that introduced by R. Feynman to the quantum electrodynamics. The diagrammatic approach suggests to

iterate the Green formula (1.48) with the unperturbed Green function taken as a zero approximation. As a result, we obtain an infinite series of iterations that should be averaged term by term. The summation of some infinite subsequence of diagrams yields an approximate equation for the average Green function $\langle \mathcal{G}(x, x'; k) \rangle$.

In these Lectures we will average the Green formula (1.48) with the use of a technique developed by A. Maradudin *et al.* [5]. This technique is similar to diagrammatic but is simpler and more elegant.

1.4.1 Averaging procedure. Dyson equation

For clarity of forthcoming calculations we rewrite the integral equation (1.48) in the symbolic form:

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 \hat{V} \mathcal{G}. \tag{1.49}$$

Here we introduce the operator $\hat{V}$ of the surface scattering. The explicit (integral) action of this operator should be interpreted as: (i) write the random potential V instead of the operator $\hat{V}$; (ii) put variables adjacent to V (i.e. next to V from the left and from the right) equal to the argument of V; (iii) integrate over this argument over the interval $(-\infty, \infty)$.

Let us define two operators: (i) *the averaging operator* $\hat{P}$ which averages everything to its right over the ensemble of realizations of the random potential $V(x)$; (ii) *the fluctuation operator* $\hat{Q}$ which extracts fluctuations from everything to its right.

Obviously, any random value is a sum of its average and fluctuations. For instance,

$$\mathcal{G} = \hat{P}\mathcal{G} + \hat{Q}\mathcal{G}. \tag{1.50}$$

Therefore the both introduced operators are related as

$$\hat{P} + \hat{Q} = \mathbf{1}. \tag{1.51}$$

Recall that the symbol $\mathbf{1}$ stands for the unit operator. In this case the unit operator is simply a number.

Since the unperturbed Green function $\mathcal{G}_0$ is deterministic (not random), we have

$$\hat{P}\mathcal{G}_0 = \mathcal{G}_0, \qquad \hat{Q}\mathcal{G}_0 = 0. \tag{1.52}$$

So, we need to seek the equation for the average Green function $\langle \mathcal{G} \rangle \equiv \hat{P}\mathcal{G}$.

Let us act separately on the both sides of the Green formula (1.49) by the averaging $\hat{P}$ and the fluctuation $\hat{Q}$ operators. Then we use the relation (1.50) and equalities (1.52) on the r.h.s. of the obtained equations. As a result, we get

$$\hat{P}\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 \hat{P}\hat{V}(\hat{P}\mathcal{G} + \hat{Q}\mathcal{G}), \tag{1.53}$$

$$\hat{Q}\mathcal{G} = \mathcal{G}_0 \hat{Q}\hat{V}(\hat{P}\mathcal{G} + \hat{Q}\mathcal{G}). \tag{1.54}$$

So, we have come to a set of two algebraic operator equations that determine two unknown functions $\hat{P}\mathcal{G}$ and $\hat{Q}\mathcal{G}$.

We now express the function $\hat{Q}\mathcal{G}$ in terms of $\hat{P}\mathcal{G}$ using the second equation (1.54):

$$\hat{Q}\mathcal{G} = (1 - \mathcal{G}_0\hat{Q}\hat{V})^{-1}\mathcal{G}_0\hat{Q}\hat{V}\hat{P}\mathcal{G}. \tag{1.55}$$

We next substitute the expression (1.55) into the first equation (1.53). After some identical operator transformations we obtain for the average Green function $\langle\mathcal{G}\rangle \equiv \hat{P}\mathcal{G}$ the operator equation which is known as *the Dyson equation*:

$$\hat{P}\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0\hat{M}\hat{P}\mathcal{G}. \tag{1.56}$$

Here we introduce the new operator $\hat{M}$ that enters the Dyson equation (1.56) as *the self-energy operator*. It is equal to

$$\begin{aligned} \hat{M} &= \hat{P}\hat{V}\left[1 + (1 - \mathcal{G}_0\hat{Q}\hat{V})^{-1}\mathcal{G}_0\hat{Q}\hat{V}\right] = \\ &= \hat{P}\hat{V}\left[1 - (1 - \mathcal{G}_0\hat{Q}\hat{V})^{-1}(1 - \mathcal{G}_0\hat{Q}\hat{V} - 1)\right] = \\ &= \hat{P}\hat{V}(1 - \mathcal{G}_0\hat{Q}\hat{V})^{-1}. \end{aligned} \tag{1.57}$$

We see that this is exactly the self-energy operator $\hat{M}$ that describes the electron interaction with the random potential $\hat{V}$.

Notice that the Dyson equation (1.56) with the self-energy (1.57) is the exact general equation for the average Green function $\langle\mathcal{G}\rangle \equiv \hat{P}\mathcal{G}$. However we cannot solve it because it is unlikely to rewrite the symbolic representation (1.57) for the self-energy operator $\hat{M}$ in a finite analytical (integral) form. Therefore, we have to simplify the self-energy $\hat{M}$.

Let us consider electron scattering at the random potential $\hat{V}$ as a *weak* scattering, in a sense. The conditions of the *weak scattering approximation* will be formulated below. In other words, we assume the scattering operator $\hat{V}$ to be a small perturbation in the expression (1.57). This assumption allows to expand the exact formula (1.57) for the self-energy $\hat{M}$ in powers of the scattering operator $\hat{V}$,

$$\hat{M} \approx \hat{P}\hat{V}(1 + \mathcal{G}_0\hat{Q}\hat{V}). \tag{1.58}$$

We take into account that, according to the correlation properties (1.17), the potential $\hat{V}$ has zero average and, therefore, is equal to the fluctuating part,

$$\langle\hat{V}\rangle \equiv \hat{P}\hat{V} = 0, \qquad\qquad \hat{Q}\hat{V} = V. \tag{1.59}$$

Thus, in the first non-vanishing (quadratic) order in the scattering operator $\hat{V}$ the self-energy operator $\hat{M}$ is given by the following formula:

$$\hat{M} \approx \hat{P}\hat{V}\mathcal{G}_0\hat{V} \equiv \langle\hat{V}\mathcal{G}_0\hat{V}\rangle. \tag{1.60}$$

Note that such approximation for the self-energy $\hat{M}$ is equivalent to the so-called *Bourret approximation* [6] in the diagrammatic technique which includes only the simplest (first) term of the diagrammatic series for the self-energy. On the other hand, in the quantum scattering theory any *weak scattering approximation of the second order in the scattering potential is known as the Born approximation*. So, we can conclude that the expression (1.60) for the self-energy operator $\hat{M}$ is written within the Born approximation.

At the end of the averaging procedure, let us substitute the formula (1.60) into the general equation (1.56), whereupon the Dyson equation for the average Green function $\langle\mathcal{G}\rangle \equiv \hat{P}\mathcal{G}$ within the Born approximation takes the following symbolic form:

$$\langle\mathcal{G}\rangle = \mathcal{G}_0 + \mathcal{G}_0\langle\hat{V}\mathcal{G}_0\hat{V}\rangle\langle\mathcal{G}\rangle. \tag{1.61}$$

This form allows a simple analytical interpretation.

1.4.2 Solving Dyson equation

Let us write down the Dyson equation (1.61) for the average Green function $\langle\mathcal{G}(x,x';k)\rangle$ in an analytical form. In accordance with the definition of the random scattering operator $\hat{V}$ and the property of the statistical homogeneity (1.47) of the average Green function $\langle\mathcal{G}(x,x';k)\rangle$, the Dyson equation can be rewritten as follows:

$$\begin{aligned}\overline{\mathcal{G}}(x-x';k) &= \mathcal{G}_0(x-x';k) + \\ &+ \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, \mathcal{G}_0(x-x_1;k) M(x_1-x_2)\overline{\mathcal{G}}(x_2-x';k).\end{aligned} \tag{1.62}$$

At the same time, in the Born approximation the kernel $M(x-x')$ of the self-energy operator $\hat{M}$ reads as:

$$M(x-x') = \langle V(x)\mathcal{G}_0(x-x')V(x')\rangle = V_0^2\mathcal{W}(|x-x'|)\mathcal{G}_0(x-x'). \tag{1.63}$$

Here we have applied the correlation properties (1.17) of the random potential $V(x)$. We draw attention to the fact that the self-energy $M(x)$ is an even function of the argument x.

Owing to all functions in the Dyson equation (1.62) being dependent only on the difference of coordinates, it is effective to apply the Fourier transformations to solve this equation.

So, we will seek the average Green function $\overline{\mathcal{G}}(x-x')$ in the Fourier representation similar to Eq. (1.38) for the unperturbed Green function $\mathcal{G}_0(x-x')$:

$$\begin{aligned}\overline{\mathcal{G}}(x-x') &= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \overline{G}(k_x)\exp[ik_x(x-x')], \\ \overline{G}(k_x) &= \int_{-\infty}^{\infty} dx\, \overline{\mathcal{G}}(x)\exp(-ik_x x).\end{aligned} \tag{1.64}$$

Let us substitute the Fourier integrals (1.64) and (1.38) for the average and unperturbed Green functions in the Dyson equation (1.62). In doing so, it is more comfortable to transfer the unperturbed Green function from the r.h.s of the Dyson equation (1.62) to the l.h.s.:

$$\int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \exp[ik_x(x-x')]\left[\overline{G}(k_x) - G_0(k_x)\right] = \tag{1.65}$$

$$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \exp(ik_x x) G_0(k_x) \exp(-ik_x x_1) \times$$

$$\times \quad M(x_1 - x_2) \int_{-\infty}^{\infty} \frac{dk'_x}{2\pi} \exp(ik'_x x_2) \overline{G}(k'_x) \exp(-ik'_x x').$$

Then we need to replace the integration variable x_1 with $x_1^{new} = x_1 - x_2$ omitting the index "*new*" for simplicity:

$$\int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \exp[ik_x(x - x')] \left[\overline{G}(k_x) - G_0(k_x)\right] = \tag{1.66}$$

$$= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \exp(ik_x x) G_0(k_x) \left[\int_{-\infty}^{\infty} dx_1 \, M(x_1) \exp(-ik_x x_1)\right] \times$$

$$\times \int_{-\infty}^{\infty} \frac{dk'_x}{2\pi} \overline{G}(k'_x) \exp(-ik'_x x') \int_{-\infty}^{\infty} dx_2 \, \exp[i(k'_x - k_x)x_2).$$

It is easy to see that the expression within the square brackets on the l.h.s. of the equation (1.66) is nothing else but the Fourier transform $M_B(k_x)$ of the self-energy (1.63),

$$\begin{aligned} M(x - x') &= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} M_B(k_x) \exp[ik_x(x - x')], \\ M_B(k_x) &= \int_{-\infty}^{\infty} dx \, M(x) \exp(-ik_x x). \end{aligned} \tag{1.67}$$

At the same time, according to the representation (1.35), the integral over x_2 is equal to $2\pi\delta(k'_x - k_x)$. Therefore we take the integral over k'_x with the help of this Dirac's delta-function:

$$\int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \exp[ik_x(x - x')] \left[\overline{G}(k_x) - G_0(k_x)\right] = \tag{1.68}$$

$$= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \exp[ik_x(x - x')] G_0(k_x) M_B(k_x) \overline{G}(k_x).$$

As a result, we obtain that in the k_x-representation the Dyson equation turns out to be a simple algebraic one:

$$\overline{G}(k_x) = G_0(k_x) + G_0(k_x) M_B(k_x) \overline{G}(k_x). \tag{1.69}$$

Its solution with respect to the Fourier transform $\overline{G}(k_x)$ of the average Green function is:

$$\overline{G}(k_x) = \frac{G_0(k_x)}{1 - M_B(k_x) \, G_0(k_x)} = \frac{1}{k^2 - k_x^2 - M_B(k_x)}. \tag{1.70}$$

Here in the last expression we have used the explicit formula (1.37) for the Fourier transform $G_0(k_x)$ of the unperturbed Green function.

In the Born approximation the formulas (1.64), (1.70), (1.67) and (1.63) complete nominally the solution of the disordered problem.

1.4.3 Average Green's function

In agreement with the Fourier representation (1.64) and the expression (1.70) for the Fourier transform $\overline{G}(k_x)$ we can write down the average Green function $\overline{\mathcal{G}}(x-x';k)$ as the Fourier integral:

$$\overline{\mathcal{G}}(x-x';k) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{\exp[ik_x(x-x')]}{k^2 - k_x^2 - M_B(k_x)}. \tag{1.71}$$

This integral should be taken over residues in the way we applied to evaluate the Fourier integral (1.39) for the unperturbed Green function $\mathcal{G}_0(x-x';k)$ (see the end of the subsection 1.3.2).

First of all, we need to find the poles of the integrand in Eq. (1.71), i.e. the singularities of the Fourier transform $\overline{G}(k_x)$. This problem reduces to solving *the dispersion equation*

$$k^2 - k_x^2 - M_B(k_x) = 0. \tag{1.72}$$

Owing to the complicated dependence of the self-energy $M_B(k_x)$ on the wave number k_x we cannot solve this equation (1.72) in a finite analytic form. At the same time, when deriving the self-energy $M(x-x')$ we already assumed weak electron scattering. As a consequence, the exact expression (1.57) for the self-energy operator $\hat{M}$ was treated perturbatively and we have obtained the self-energy $M(x-x')$ as the Born (quadratic) approximation (1.63) in the random scattering potential $V(x)$. Therefore, there seems to be no other way to proceed but to solve the dispersion equation (1.72) perturbatively considering the self-energy $M_B(k_x)$ as a perturbation.

So, we solve the dispersion equation (1.72) by iterations in the small self-energy $M_B(k_x)$ which is assumed much smaller than the "energy" k^2 ($|M_B(k_x)| \ll k^2$). At the first step of iterations we neglect the self-energy $M_B(k_x)$ and get the unperturbed result $k_x = \pm k$. At the second step we include a small correction δk to the unperturbed wave number k. In other words, we seek the solution of the dispersion equation (1.72) in the form:

$$k_x = \pm(k + \delta k). \tag{1.73}$$

Obviously, we should calculate the correction δk under the same restrictions that were used for the derivation of the self-energy $M(x)$. Hence, an important point now is to formulate explicitly the *conditions of weak electron scattering*. A detailed analysis, which we do not perform here, shows that the assumed Born approximation for the self-energy $M_B(k_x)$ is valid as long as the following two inequalities hold:

$$|\delta k| \ll k, \tag{1.74}$$

$$|\delta k| R_c \ll 1. \tag{1.75}$$

These conditions mean that the deviation δk from the unperturbed wave number k must be the smallest parameter of the dimension of wave number. Indeed, so far there were just two parameters with the wave-number dimensions in our problem. Those are the electron wave number k and the inverse correlation radius R_c^{-1}. The former specifies the unperturbed electron states, while the latter characterizes the perturbation of electron states by the random scattering potential $V(x)$. So, the shift δk has to be small in

comparison with both of them. We point out that we do not assume any predetermined interrelation between the electron wave length k^{-1} and the correlation radius R_c.

Thus, we should solve the dispersion equation (1.72), i.e. find the correction δk, under the conditions (1.74) and (1.75). Notice that we have used the inequality (1.74) when we present the solution of the equation in the form (1.73).

Let us now substitute the expression (1.73) into the dispersion equation (1.72) and take into account that the Fourier transform $M_B(k_x)$ (1.67) is an even function of the argument k_x:

$$k^2 - (k + \delta k)^2 - M_B(k + \delta k) = 0. \tag{1.76}$$

Then, owing to the condition (1.74), we can expand the second term on the l.h.s of Eq. (1.76) up to linear in δk terms inclusive and neglect second and higher order terms in δk. After cancelling identical terms we get the wave number shift δk:

$$\delta k = -M_B(k + \delta k)/2k. \tag{1.77}$$

Next we would like to eliminate the correction δk from the argument of the self-energy $M_B(k + \delta k)$. To understand if we can do it we use the explicit expression for $M_B(k + \delta k)$ that stems directly from the definition (1.67) and formula (1.63):

$$M_B(k + \delta k) = V_0^2 \int_{-\infty}^{\infty} dx\, \mathcal{W}(|x|)\mathcal{G}_0(x) \exp(-ikx) \exp(-i\delta kx). \tag{1.78}$$

The first multiplier $\mathcal{W}(|x|)$ in the integrand has a maximum at $x = 0$ and monotonously decreases over the correlation radius R_c. The second factor $\mathcal{G}_0(x) \exp(-ikx)$ varies over the electron wave length k^{-1}. The variation scale δk^{-1} of the third term $\exp(-i\delta kx)$ is the longest. In other words, the function $\exp(-i\delta kx)$ is a smooth function in comparison with the rapidly oscillating exponents $\mathcal{G}_0(x) \exp(-ikx)$ and the correlation coefficient $\mathcal{W}(|x|)$ as far as the conditions (1.74) and (1.75) hold. So, we can replace the function $\exp(-i\delta kx)$ by one. This means that the deviation δk can safely be removed from the argument of the self-energy $M_B(k + \delta k)$ in the equation (1.77) as far as the weak scattering limit (1.74) and (1.75) is concerned.

As a result of the above simplifications, we come to the following expression for the wave-number shift δk:

$$\delta k = -M_B(k)/2k. \tag{1.79}$$

Thus, within the Born approximation (1.74) and (1.75) the poles of the Fourier transform $\overline{G}(k_x)$ of the average Green function are

$$k_x = \pm(k - M_B(k)/2k). \tag{1.80}$$

Let us now recall that we are evaluating the integral over k_x in the equation (1.71) to obtain the explicit formula for the average Green function $\overline{\mathcal{G}}(x - x'; k)$. We have found the poles (1.80) of the integrand. A next step should be taking the integral over residues as it was done for the unperturbed Green function $\mathcal{G}_0(x - x'; k)$ (see the end of the subsection 1.3.2). Fortunately, we do not need to do this. To write down an answer it is sufficient to compare the expressions (1.40) and (1.80) for the poles of the unperturbed $G_0(k_x)$ and average $\overline{G}(k_x)$ Green functions. The comparison leads to the conclusion: *In*

the Born approximation (1.74) and (1.75) *the average Green function* $\overline{\mathcal{G}}(x-x';k)$ *is always equal to the unperturbed Green function* $\mathcal{G}_0(x-x')$ *with the wave number* k *reduced by the value* $M_B(k)/2k$:

$$\overline{\mathcal{G}}(x-x';k) = \mathcal{G}_0(x-x';k-M_B(k)/2k). \tag{1.81}$$

It is necessary to emphasize that this conclusion is valid if the imaginary part of the self-energy $M_B(k)$ is negative or zero. This is the only case when the poles of both Green functions are similar to each other. However, such limitation on the imaginary part of $M_B(k)$ should not be considered strict because in systems with no energy sources the non-positiveness of $M_B(k)$ is dictated by the energy conservation law (see, e.g., [1, 7]).

We will see below that the self-energy $M_B(k)$ is complex. Basing on this fact, let us extract the real and the imaginary part of the correction δk:

$$\delta k = \gamma + i/2L_{ts}; \tag{1.82}$$

$$\gamma = -\Re M_B(k)/2k, \tag{1.83}$$

$$1/L_{ts} = -\Im M_B(k)/k, \qquad L_{ts} > 0. \tag{1.84}$$

Hereinafter the symbols "$\Re$" and "$\Im$" stand for the real and imaginary parts respectively.

After the lengthy algebra we are in a position now to write down the explicit formula for *the average Green function* $\overline{\mathcal{G}}(x-x';k)$ *of a 1D disordered conductor*. According to the conclusion (1.81) and the equation (1.27) for the unperturbed Green function $\mathcal{G}_0(x-x';k)$ using the notation (1.83) and (1.84) we get:

$$\overline{\mathcal{G}}(x-x';k) = \frac{\exp[i(k+\gamma)|x-x'|]}{2ik}\exp(-\frac{|x-x'|}{2L_{ts}}). \tag{1.85}$$

Brief analysis of the expression (1.85) for the average Green function of a 1D disordered conductor: Electron scattering by a random potential gives rise to the complex deviation δk of the electron wave number from the unperturbed value k. This fact causes the phase renormalization and attenuation of the average Green function along the length of the conductor. The real part γ of the complex shift δk is responsible for the disorder-induced modification to the phase of the Green function. The quantity L_{ts} which is specified by the imaginary part of δk has the meaning of the attenuation length.

1.4.4 Electron mean free path

We now need to derive explicit expressions for the phase modification γ and the attenuation length L_{ts} to associate them with the properties of the random scattering potential $V(x)$. According to definitions (1.83) and (1.84) this problem reduces to the derivation of the explicit expression for the self-energy $M_B(k)$.

Let us substitute the equation (1.63) for $M(x)$ into the formula (1.67) at $k_x = k$:

$$M_B(k) = V_0^2\int_{-\infty}^{\infty} dx\,\mathcal{W}(|x|)\mathcal{G}_0(x)\exp(-ikx). \tag{1.86}$$

Then we apply the Fourier representation (1.38) for the unperturbed Green function $\mathcal{G}_0(x)$:

$$M_B(k) = V_0^2 \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, G_0(k_x) \int_{-\infty}^{\infty} dx\, \mathcal{W}(|x|) \exp[-i(k-k_x)x]. \tag{1.87}$$

We next take into account that the integral over x is nothing else but the Fourier transform of the correlation coefficient $\mathcal{W}(|x|)$:

$$\begin{aligned} \mathcal{W}(|x|) &= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, W(k_x) \exp(ik_x x), \\ W(k_x) &= \int_{-\infty}^{\infty} dx\, \mathcal{W}(|x|) \exp(-ik_x x). \end{aligned} \tag{1.88}$$

So, the self-energy $M_B(k)$ takes the form:

$$M_B(k) = V_0^2 \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, W(k-k_x) G_0(k_x). \tag{1.89}$$

At this step of our calculations let us substitute the explicit expression (1.37) for the Fourier transform $G_0(k_x)$ of the unperturbed Green function $\mathcal{G}_0(x)$ into Eq. (1.89). In this way we arrive at the result:

$$M_B(k) = V_0^2 \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \frac{W(k-k_x)}{k^2 - k_x^2 + i0}. \tag{1.90}$$

We now should present the self-energy $M_B(k)$ as an explicit complex value. In other words, we need to extract the real and the imaginary parts from the equation (1.90). To this end we divide the Fourier transform $G_0(k_x)$ into two terms:

$$G_0(k_x) = \frac{1}{k^2 - k_x^2 + i0} = \frac{1}{2k} \left(\frac{1}{k - k_x + i0} + \frac{1}{k + k_x + i0} \right). \tag{1.91}$$

After that we use the following Dirac identity:

$$\frac{1}{k - k_x \pm i0} = \mathcal{P}.\mathcal{V}.\frac{1}{k - k_x} \mp \pi i \delta(k - k_x). \tag{1.92}$$

Here the symbol $\mathcal{P}.\mathcal{V}.$ stands for a principal value of an integral. According to the identity (1.92), the Fourier transform $G_0(k_x)$ is given by

$$\begin{aligned} G_0(k_x) &= \frac{1}{k^2 - k_x^2 + i0} = \\ &= \mathcal{P}.\mathcal{V}.\frac{1}{k^2 - k_x^2} - \frac{\pi i}{2k} \left[\delta(k - k_x) + \delta(k + k_x) \right]. \end{aligned} \tag{1.93}$$

We next substitute this expression (1.93) into the formula (1.90) and take some integrals with the help of the Dirac delta-functions. At the end we find out that the self-energy is described by

$$M_B(k) = V_0^2\, \mathcal{P}.\mathcal{V}. \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \frac{W(k-k_x)}{k^2 - k_x^2} - i \frac{V_0^2}{4k} \left[W(0) + W(2k) \right]. \tag{1.94}$$

It is necessary to emphasize that the evenness of the correlation coefficient $\mathcal{W}(|x|)$ implies the reality of the Fourier transform $W(k_x)$. Therefore, in the representation (1.94) the

first term is the real part and the second one is the imaginary part of the self-energy $M_B(k)$.

Thus, in accordance with the definitions (1.83) and (1.84), the explicit expressions for the phase modification γ and the attenuation length L_{ts} take the following ultimate forms:

$$\gamma = -\frac{V_0^2}{2k}\,\mathcal{P}.\mathcal{V}.\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,\frac{W(k-k_x)}{k^2-k_x^2}. \tag{1.95}$$

$$\frac{1}{L_{ts}} = \frac{V_0^2}{4k^2}\left[W(0)+W(2k)\right], \qquad L_{ts}>0. \tag{1.96}$$

The real spectrum shift γ does not play any role in the conducting properties of 1D disordered electron systems. At the same time, the imaginary part $1/2L_{ts}$ of the spectrum deviation δk changes these properties drastically. Nothing else but the imaginary part $1/2L_{ts}$ forms the conductance and causes the effect of the strong localization. Therefore, hereinafter we will analyze only the attenuation length L_{ts} of the average Green function of a 1D disordered conductor.

It is a quite general result of the quantum scattering theory that *the attenuation length* L_{ts} of the average single-particle Green function *is just the "outgoing" mean free path of electrons* which is formed by scattering from a given state into all possible states (including the given one). Indeed, the attenuation length L_{ts} is originated from the imaginary part of the self-energy (see Eq. (1.84), where $L_{ts}^{-1} = -\Im M_B(k)/k$). On the other hand, the imaginary part of the self-energy $M_B(k)$ is proportional to the outgoing term of the quantum integral of collisions. Since *the outgoing term is proportional to the full cross section of scattering*, the corresponding *outgoing mean free path is inversely proportional to the full cross section.* Therefore it is often referred to as *the full (total) mean free path.*

We know that the full cross section of scattering is proportional to the probability of electron scattering from a given state into all possible states (including the given one). *In the 1D case an electron* has only two possibilities for scattering: It *can be scattered either forward or backward only.* Let us recall that we consider an elastic scattering because the random potential $V(x)$ is time-independent and hence the Schrödinger equation has stationary (with well-defined energy) solutions. Elastic scattering does not change the initial energy (and hence the modulus of the wave vector, i.e. the wave number k) of an electron. It changes only the direction of the electron motion, i.e. only the sign of the velocity (the sign of the wave vector $\vec{k}$). Therefore, *the modulus of the modification* $\Delta\vec{k}$ *of the electron wave vector is equal to zero* ($|\Delta\vec{k}|=0$) *at the forward elastic scattering and is equal to the double initial wave number* ($|\Delta\vec{k}|=2k$) *at the backward elastic scattering.* Let us now recall that the scattering probability is a function of the modification $|\Delta\vec{k}|$ of the electron wave vector by a scattering process.

In agreement with the above statements, we can conclude that the first term in the formula (1.96) is proportional to the probability of the forward electron scattering and the second one is proportional to the probability of the backward electron scattering. Based on this conclusion, the full electron mean free path can be presented as

$$L_{ts}^{-1} = L_{fs}^{-1} + L_{bs}^{-1}\,;$$

$$L_{fs}^{-1} = \frac{V_0^2}{4k^2} W(0), \qquad L_{bs}^{-1} = \frac{V_0^2}{4k^2} W(2k). \tag{1.97}$$

In this expression the length L_{fs} is related to the forward electron scattering, while the length L_{bs} is associated with the backward scattering. Therefore, *the length L_{fs} is* called *the mean free path of forward electron scattering* and *the length L_{bs}* is referred to as *the backscattering mean free path.*

Usually, the backscattering length L_{bs} is larger than the length L_{fs} because $W(2k) < W(0)$. So, $L_{ts}^{-1} \approx L_{fs}^{-1}$ and the average Green function attenuates along the conductor mainly on the scale of the mean free path of forward electron scattering L_{fs}. However, in the second part of the Lectures we will see that the conductance of 1D disordered conductors is completely determined by the backscattering length L_{bs} alone but *not* by L_{fs}.

Concluding this subsection we would like to point out that, owing to the Born restrictions (1.74) and (1.75), the outgoing mean free path L_{ts} of electrons far exceeds both the electron wave length k^{-1} and the correlation radius R_c:

$$kL_{ts} \gg 1, \qquad L_{ts} \gg R_c. \tag{1.98}$$

It is clear that the first requirement allows to think of electrons as "quasifree" (but not quasiclassical!) particles. The second inequality is, in fact, the necessary and sufficient condition for *the statistical approach* to the problem of electron interaction with a random potential.

Chapter 2

Conductance of 1D Disordered Conductor

The previous Lectures discussed the spectral properties of disordered 1D electron systems. To this end we derived and analyzed *the single-particle average Green function*. Now we are starting to consider conducting (transport) properties of such systems. More specifically, we will obtain and study the conductance of disordered 1D electron systems. The main effects which we expect to find out are the ballistic and strong-localization regimes of conduction. It is necessary to emphasize that the strong localization is caused by coherent multiple scattering of electrons by a random potential. Presently the complete self-consistent theory of these phenomena exists only in a 1D case.

2.1 Conductance-Problem Formulation

We consider *a conducting 1D disordered wire of finite length* L. Let the x axis be directed along a straight line of the wire which occupies the following spatial interval:

$$-L/2 \leq x \leq L/2. \tag{2.1}$$

Note that in our discussion devoted to the average Green function we treated an infinitely long disordered 1D wire. It is important to point out here that the localization makes the conductance of an infinite 1D wire to vanish. So, it is sensible to consider a 1D conductor of finite length L in order to analyze both the ballistic and localization regimes as well as the crossover between them.

As before, we choose the free electron model, i.e. the dispersion law of electrons is assumed to be quadratic and isotropic (1.16).

For convenience of further calculations, we introduce *the dimensionless conductance* $T(L)$ of the system. It is defined as the conductance $G(L)$ of the wire divided by the conductance quantum $e^2/\pi\hbar$:

$$T(L) = \frac{G(L)}{e^2/\pi\hbar}. \tag{2.2}$$

Here e is the elementary charge. Below we will see that the dimensionless conductance $T(L)$ has an independent physical meaning.

In accordance with the standard linear response theory by R. Kubo [8], the conductance (as well as conductivity) of a non-interacting system is expressed via the product of differences between the advanced and retarded single-electron Green functions (see, e.g., papers [9, 10] and the book [11]). It has been proved [12, 13] that under the conditions of weak electron scattering one can neglect the products of the identical Green functions (both retarded and both advanced) in the general expression for the conductance. Then, taking into account that the advanced Green function transforms into the retarded Green function by simultaneous complex conjugation and swapping the arguments, the dimensionless conductance $T(L)$ at zero temperature can be presented as

$$T(L) = -\frac{4}{L^2} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \, \frac{\partial \mathcal{G}(x, x'; k_F)}{\partial x} \, \frac{\partial \mathcal{G}^*(x, x'; k_F)}{\partial x'} \, . \tag{2.3}$$

The asterisk "$*$" stands for complex conjugation. Note that the Fermi wave number k_F replaces the wave number k in this and all further equations because the electron gas is assumed degenerate, i.e. obeying the Fermi-Dirac statistics.

The retarded single-electron Green function $\mathcal{G}(x, x'; k_F)$ was introduced in the previous Lectures. For completeness of the problem statement we rewrite here the equation for the retarded Green function of a disordered 1D conductor:

$$\left(\frac{d^2}{dx^2} + k_F^2 + i0 \right) \mathcal{G}(x, x') - V(x) \mathcal{G}(x, x') = \delta(x - x'). \tag{2.4}$$

Hereinafter the symbol "$i0$" means an arbitrarily small, tending to zero, positive imaginary quantity. As before, the Green function satisfies *the radiative boundary conditions* at the wire ends $x = \pm L/2$. In the next section we will give the explicit expressions for the radiative boundary conditions in the case of a conductor of finite length L.

To describe the electron scattering we have introduced *the random potential* $V(x)$ in the equation (2.4), which specifies the inhomogeneity (*disorder*) of the problem. Its correlation properties are defined by the formula (1.17) from the previous part of the Lectures.

We managed to solve the problem of the average Green function under the conditions of *weak electron scattering* only, i.e. within *the Born approximation*. The same conditions of the weakness of the random potential $V(x)$ will be assumed here in deriving the conductance. Since the real spectrum shift γ obtained in the previous Lectures does not contribute to the conductance, the weak-scattering restrictions should be formulated for the electron total (outgoing) mean free path L_{ts}. As we found out earlier, the electron relaxation length L_{ts} must be large in comparison with the "microscopic" lengths of our problem, namely, with the electron wavelength k_F^{-1} and the correlation radius R_c: $k_F L_{ts} \gg 1$ and $L_{ts} \gg R_c$. Moreover, to make averaging over the realizations of the random potential $V(x)$ meaningful we need to assume similar requirements for the conductor length L, i.e., $k_F L \gg 1$ and $L \gg R_c$. All these conditions can be collected into the following inequality:

$$\max\{k_F^{-1}, R_c\} \ll \min\{L_{ts}, L\}. \tag{2.5}$$

We point out that we do not assume any predetermined interrelations between the Fermi wavelength k_F^{-1} and the correlation radius R_c or between the wire length L and the total relaxation length L_{ts}.

Our goal is to calculate the dimensionless conductance $T(L)$ (2.2) averaged over the ensemble of realizations of the random potential $V(x)$. According to the Kubo formula (2.3), this problem is reduced to averaging the product of two retarded Green's functions, i.e., to searching the average two-particle Green function. It is very important to underline that for the 1D case this procedure can not be performed by an approximate method of solving the Dyson equation which we used to calculate the average single-particle Green function. Therefore, below we will apply one of more rigorous methods that allow correct solution of 1D two-particle quantum problems with disorder. We mention that those methods give the same result for the average single-particle Green function which can be derived by the traditional procedure of solving the Dyson equation.

2.2 Two-scale Model

According to the weak-scattering assumption (2.5), there exist two groups of substantially different spatial scales in our problem. On the one hand, there is a group of "macroscopic" lengths: the wire length L and the relaxation length L_{ts}. On the other hand, there is a pair of "microscopic" lengths: the electron wavelength k_F^{-1} and the correlation radius R_c. The existence of two scales allows to apply *the two-scale model of oscillations* for treating the equation (2.4) for the retarded Green function $\mathcal{G}(x, x')$.

We start with the well-known representation for the 1D Green function $\mathcal{G}(x, x')$ (see Eq. (1.28) from the previous Lectures):

$$\mathcal{G}(x, x') = W^{-1}[\psi_+(x)\psi_-(x')\Theta(x - x') + \psi_+(x')\psi_-(x)\Theta(x' - x)] \ . \tag{2.6}$$

Recall that in this representation the functions $\psi_\pm(x)$ are two linearly independent solutions of the corresponding homogeneous equation, i.e. equation (2.4) with zero instead of the delta-function on the r.h.s. The solution $\psi_-(x)$ satisfies the radiative boundary condition at the left open end $x = -L/2$ and the solution $\psi_+(x)$ satisfies the radiation boundary condition at the right open end $x = +L/2$ of the wire. The Wronskian W of the functions $\psi_\pm(x)$ is defined as usual,

$$W = \psi_-(x')\psi'_+(x') - \psi_+(x')\psi'_-(x'). \tag{2.7}$$

Hereinafter a prime at a function stands for the derivative of this function with respect to its argument. Note that the Wronskian W of any Hermitian problem does not depend on the coordinate x'. Therefore, we can calculate it at any convenient point within the interval (2.1). The symbol $\Theta(x)$ in Eq. (2.6) denotes the Heaviside unit-step Θ-function.

We know that two linearly independent solutions $\psi_\pm^{(0)}(x)$ of the unperturbed (when $V(x) = 0$) boundary-value problem are the plane waves:

$$\psi_\pm^{(0)}(x) = \exp(\pm ik_F x), \qquad k_F \to k_F + i0. \tag{2.8}$$

The plane wave $\psi_+^{(0)}(x) = \exp(ik_F x)$ propagates to the right and the plane wave $\psi_-(x)^{(0)} = \exp(-ik_F x)$ propagates to the left. It is clear that in the 1D perturbed problem these "initial" electron waves can be scattered by the random potential $V(x)$ both forward and backward. As a result, the electron state will be made up by the sum of transmitted and reflected waves. Therefore, we can seek the functions $\psi_\pm(x)$ as superposition of modulated waves propagating in opposite directions along the x-axis,

$$\psi_{\pm}(x) = \pi_{\pm}(x)\exp(\pm ik_F x) - i\gamma_{\pm}(x)\exp(\mp ik_F x). \tag{2.9}$$

The radiative boundary conditions for the wave functions $\psi_{\pm}(x)$ imply the absence of waves scattered by the wire ends $x = \pm L/2$. It means that the perturbed wave functions $\psi_{\pm}(x)$ have to transform into the respective plane waves $\psi_{\pm}^{(0)}(x)$ at the ends $x = \pm L/2$,

$$\psi_{\pm}(\pm L/2) = \exp(\pm ik_F x). \tag{2.10}$$

In this way the radiation boundary conditions (2.10) for the functions $\psi_{\pm}(x)$ lead to the "initial" conditions for the amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$:

$$\pi_{\pm}(\pm L/2) = 1, \qquad \gamma_{\pm}(\pm L/2) = 0. \tag{2.11}$$

Obviously, if the random scattering potential $V(x)$ tends to zero, the introduced amplitudes $\pi_{\pm}(x)$ tend to unity and $\gamma_{\pm}(x)$ tend to zero as well.

It is noteworthy that scales of variation of the amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$ in Eq. (2.9) as functions of the coordinate x are macroscopic: they are determined by either the relaxation length L_{ts} or the wire length L. Thus, within the two-scale approximation (2.5) they are smooth functions of the coordinate x in comparison with the rapidly oscillating exponents $\exp(\pm ik_F x)$ and the correlation coefficient $\mathcal{W}(x)$.

2.2.1 Dynamic equations for smooth amplitudes

According to the representations (2.6) and (2.9), the problem of calculating the Green function $\mathcal{G}(x, x')$ is reduced to finding the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$. Therefore, first of all, we should derive equations for these functions. Under the weak-scattering condition (2.5) the appropriate equations are deduced by a standard *method of averaging over rapid phases* (see, e.g., the book [14]). Let us now apply that method to our problem.

We start from the homogeneous equation for the functions $\psi_{\pm}(x)$. This equation corresponds to the equation (2.4) for the Green function $\mathcal{G}(x, x')$ but has zero instead of the delta-function on the r.h.s.:

$$\left(\frac{d^2}{dx^2} + k_F^2\right)\psi_{\pm}(x) - V(x)\psi_{\pm}(x) = 0. \tag{2.12}$$

We need to calculate explicitly the second derivative of $\psi_{\pm}(x)$ with respect to x. The first derivative of $\psi_{\pm}(x)$ are

$$\begin{aligned} \frac{d}{dx}\psi_{\pm}(x) &= \pi'_{\pm}(x)\exp(\pm ik_F x) - i\gamma'_{\pm}(x)\exp(\mp ik_F x) \pm \\ &\pm ik_F\pi_{\pm}(x)\exp(\pm ik_F x) \mp k_F\gamma_{\pm}(x)\exp(\mp ik_F x), \end{aligned} \tag{2.13}$$

whence we get the second derivative:

$$\frac{d^2}{dx^2}\psi_{\pm}(x) = \pi''_{\pm}(x)\exp(\pm ik_F x) - i\gamma''_{\pm}(x)\exp(\mp ik_F x) \pm$$

$$\begin{aligned} \pm \quad & 2ik_F\pi'_\pm(x)\exp(\pm ik_Fx) \mp 2k_F\gamma'_\pm(x)\exp(\mp ik_Fx) - \\ - \quad & k_F^2\pi_\pm(x)\exp(\pm ik_Fx) + ik_F^2\gamma_\pm(x)\exp(\mp ik_Fx). \end{aligned} \tag{2.14}$$

We next take into account the following estimations for the derivatives of the smooth amplitudes $\pi_\pm(x)$ and $\gamma_\pm(x)$. Since a derivative of a function can be roughly estimated as the value of the function itself divided by a scale of its variation, we have:

$$\begin{aligned} &\pi'_\pm(x) \sim \pi_\pm/L_{ts} \ \text{ or } \ \sim \pi_\pm/L, \qquad && \pi''_\pm(x) \sim \pi_\pm/L_{ts}^2 \ \text{ or } \ \sim \pi_\pm/L^2; \\ &\gamma'_\pm(x) \sim \gamma_\pm/L_{ts} \ \text{ or } \ \sim \gamma_\pm/L, \qquad && \gamma''_\pm(x) \sim \gamma_\pm/L_{ts}^2 \ \text{ or } \ \sim \gamma_\pm/L^2. \end{aligned} \tag{2.15}$$

In accordance with these estimations we can neglect the first two terms containing the second derivatives of the smooth amplitudes $\pi_\pm(x)$ and $\gamma_\pm(x)$ in the exact equation (2.14). So, within the two-scale model (2.5) the second derivative of $\psi_\pm(x)$ looks like

$$\frac{d^2}{dx^2}\psi_\pm(x) \approx \pm 2ik_F\pi'_\pm(x)\exp(\pm ik_Fx) \mp 2k_F\gamma'_\pm(x)\exp(\mp ik_Fx) - k_F^2\psi_\pm(x). \tag{2.16}$$

Here we have used the explicit expression (2.9) for the wave functions $\psi_\pm(x)$.

We now substitute the formulas (2.16) and (2.9) into the equation (2.12) and cancel identical terms proportional to the "Fermi energy" k_F^2. As a result, we come to the equations:

$$\begin{aligned} & \pm 2ik_F\pi'_\pm(x)\exp(\pm ik_Fx) \mp 2k_F\gamma'_\pm(x)\exp(\mp ik_Fx) \quad - \\ -\pi_\pm(x)V(x)\exp(\pm ik_Fx) + i\gamma_\pm(x)V(x)\exp(\mp ik_Fx) \quad &= \quad 0. \end{aligned} \tag{2.17}$$

So, we have got only two equations with respect to four unknown functions $\pi_\pm(x)$ and $\gamma_\pm(x)$. Moreover, these equations contain functions which vary over both scales of our problem. Indeed, the smooth amplitudes $\pi_\pm(x)$ and $\gamma_\pm(x)$ vary over the macroscopic lengths (either the wire length L or the mean free path L_{ts}) while the random potential $V(x)$ and the exponents $\exp(\pm ik_Fx)$ vary over the microscopic lengths: the correlation radius R_c and the electron wavelength k_F^{-1} respectively. Obviously, we need four instead of just two equations. Moreover, in those four equations sought-for all terms should have only macroscopic variation scales. To deduce the required four equations we will average our two equations (2.17) over the microscopic lengths k_F^{-1} and R_c.

The averaging procedure consists of the following steps:

(i) First, we multiply both sides of the equations (2.17) from the left through by the exponent $\exp(\mp ik_Fx)$.

(ii) Then we replace the variable x with x' and integrate the obtained equations obtained term by term over x' within the spatial interval $(x-l, x+l)$. At the same time we divide them, also term by term, by the interval length $2l$. In other words, at this step of averaging we act on the both sides of the equations from the left through by the integral operator:

$$\int_{x-l}^{x+l} \frac{dx'}{2l} \dots . \tag{2.18}$$

To eliminate the microscopic scales k_F^{-1} and R_c but retain the macroscopic scales L and L_{ts} the length $2l$ of the averaging interval must be intermediate between the above-introduced microscopic and macroscopic scales. This means that the length l must be much larger than k_F^{-1} and R_c while much smaller than L and L_{ts}:

$$\max\{k_F^{-1}, R_c\} \ll l \ll \min\{L_{ts}, L\}. \tag{2.19}$$

Within the domain (2.19) the length l can be varied arbitrarily.

(iii) Under the conditions (2.19) the smooth amplitudes $\pi_\pm(x)$ and $\gamma_\pm(x)$ should be taken outside the integral operator (2.18) at the point $x' = x$.

(iv) In this way we arrive at two equations for the four unknown functions $\pi_\pm(x)$ and $\gamma_\pm(x)$. To deduce the other pair of the equations we need to repeat all the averaging operations (i) – (iii) using the multiplier $\exp(\pm ik_F x)$ instead of the exponent $\exp(\mp ik_F x)$.

At the end of the averaging procedure we obtain the set of *four first order differential dynamic (causal) equations* with variable coefficients:

$$\begin{aligned} \pi'_\pm(x) \pm i\eta(x)\pi_\pm(x) \pm \zeta^*_\pm(x)\gamma_\pm(x) = 0 \ , \\ \gamma'_\pm(x) \mp i\eta(x)\gamma_\pm(x) \pm \zeta_\pm(x)\pi_\pm(x) = 0 \ . \end{aligned} \tag{2.20}$$

These equations are complemented by *the four initial conditions* (2.11).

In the equations (2.20) the variable coefficients $\eta(x)$ and $\zeta_\pm(x)$ are defined by the expressions

$$\begin{aligned} \eta(x) &= \frac{1}{2k_F}\int_{x-l}^{x+l}\frac{dx'}{2l}\exp(\mp ik_F x')V(x')\exp(\pm ik_F x') = \\ &= \frac{1}{2k_F}\int_{x-l}^{x+l}\frac{dx'}{2l}V(x'), \\ \zeta_\pm(x) &= \frac{1}{2k_F}\int_{x-l}^{x+l}\frac{dx'}{2l}\exp(\pm ik_F x')V(x')\exp(\pm ik_F x') = \\ &= \frac{1}{2k_F}\int_{x-l}^{x+l}\frac{dx'}{2l}\exp(\pm 2ik_F x')V(x'). \end{aligned} \tag{2.21}$$

According to these definitions *the coefficients $\eta(x)$ and $\zeta_\pm(x)$ represent the space-averaged random fields associated with the scattering potential $V(x)$ of the electron-disorder interaction.* Since the random potential $V(x)$ is real, the function $\eta(x)$ is real too. At the same time the random fields $\zeta_\pm(x)$ are complex conjugate to each other,

$$\zeta^*_\pm(x) = \zeta_\mp(x). \tag{2.22}$$

Thus, *we have come from the boundary-value problem* (2.4) for the Green function $\mathcal{G}(x, x')$ *to the dynamic problem* (2.20) with the initial conditions (2.11) for the smooth amplitudes $\pi_\pm(x)$ and $\gamma_\pm(x)$. The coordinate x plays the role of time in our dynamic problem.

2.2.2 Correlation relations for the random fields

It follows from the dynamic equations (2.20) that the smooth amplitudes $\pi_\pm(x)$ and $\gamma_\pm(x)$ are causal functionals of the space-averaged random fields $\eta(x)$ and $\zeta_\pm(x)$. Since we are interested in quantities averaged over realizations of the random potential $V(x)$, statistical properties of the fields $\eta(x)$ and $\zeta_\pm(x)$ become crucial.

As long as the average of the scattering potential $V(x)$ is equal to zero (see Eq. (1.17)), *the random fields $\eta(x)$ and $\zeta_\pm(x)$ are also zero-averaged* according to the definitions (2.21),

$$\langle \eta(x) \rangle = \langle \zeta_\pm(x) \rangle = 0. \tag{2.23}$$

From this equation and the weak scattering conditions (2.5) it follows that all the fields $\eta(x)$ and $\zeta_\pm(x)$ can be considered as Gaussian random processes whose properties are completely specified by their binary correlators. We will calculate the correlators at the end of this subsection and here we give and discuss just the ultimate expressions for them:

$$\begin{gathered} \langle \eta(x)\zeta_\pm(x') \rangle = \langle \zeta_\pm(x)\zeta_\pm(x') \rangle = 0; \\ \langle \eta(x)\eta(x') \rangle = L_{fs}^{-1}\delta(x-x'), \qquad \langle \zeta_\pm(x)\zeta_\pm^*(x') \rangle = L_{bs}^{-1}\delta(x-x') \quad . \end{gathered} \tag{2.24}$$

It is essential that the both known electron-relaxation lengths (1.97), L_{fs} and L_{bs}, emerge in the correlators (2.24) of the random fields $\eta(x)$ and $\zeta_\pm(x)$. They are given by the expressions:

$$L_{fs}^{-1} = \frac{V_0^2}{4k_F^2} W(0), \tag{2.25}$$

$$L_{bs}^{-1} = \frac{V_0^2}{4k_F^2} W(2k_F). \tag{2.26}$$

The function $W(k_x)$ is the Fourier transform of the correlation coefficient $\mathcal{W}(|x|)$ from Eq. (1.17),

$$\begin{aligned} \mathcal{W}(|x|) &= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, W(k_x) \exp(ik_x x), \\ W(k_x) &= \int_{-L/2}^{L/2} dx\, \mathcal{W}(|x|) \exp(-ik_x x). \end{aligned} \tag{2.27}$$

Note that now the integration over the coordinate x is performed within the finite interval (2.1) instead of the infinite one $(-\infty, \infty)$. This is precisely the interval occupied by the disordered 1D wire of the finite length L.

We know from the previous Lectures that the length L_{fs} is related to the forward electron scattering (i.e. without changing the sign of the electron velocity), while L_{bs} is associated with the backward scattering. In our consideration the length L_{fs} specifies the correlator $\langle \eta(x)\eta(x') \rangle$ whereas the backscattering length L_{bs} controls the correlator $\langle \zeta_\pm(x)\zeta_\pm^*(x') \rangle$. Hence we can conclude that *the real random field $\eta(x)$ is responsible for the*

forward electron scattering. At the same time, *the complex random fields* $\zeta_{\pm}(x)$ *describe the backscattering of electrons.*

Recall that the superposition of the inverse lengths (2.25) and (2.26) is the inverse outgoing length of attenuation of the average Green function $\langle \mathcal{G}(x,x';k_F)\rangle = \overline{\mathcal{G}}(x-x';k_F)$,

$$L_{ts}^{-1} = L_{fs}^{-1} + L_{bs}^{-1}. \tag{2.28}$$

This electron outgoing mean free path is exactly the scattering length that enters the weak scattering condition (2.5).

In consequence of the correlation properties (2.23) and (2.24) *the random fields* $\eta(x)$ *and* $\zeta_{\pm}(x)$ *turn out to be* statistically independent of each other and delta-correlated. Therefore we can regard them as *real or complex, respectively, Gaussian "white noise".* We emphasize that this is a general result for a weak overbarrier scattering. This fact plays a dominating role in averaging procedures. Strictly speaking, the delta-functions are not exact in the expressions (2.24). As we will see below, the correlators we deal with are sharp (of the order of the space averaging scale l (2.19)) functions of the coordinate difference $x-x'$. Recall that in the equations (2.20) for the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$ all the terms vary essentially only over macroscopic scales (either the wire length L or the mean free path L_{ts}). Thus, due to conditions (2.19) such correlators should be replaced with the corresponding delta-functions.

Useful integrals

Before calculating the binary correlators (2.24) we need to take and analyze some useful integrals that will appear in the calculations.

(i) A first useful integral is

$$\int_{x-l}^{x+l} \frac{dx_1}{2l} \exp[\pm ik_x(x_1-x)] = \int_{-l}^{l} \frac{dx_1}{2l} \exp(\pm ik_x x_1) = \frac{\sin(k_x l)}{k_x l}. \tag{2.29}$$

This integral has the highest (absolute) maximum at $k_x = 0$ as a function of the wave number k_x and decreases towards the both sides of the maximum with the typical scale $\Delta k_x \sim l^{-1}$. So, this integral is a sharp function in comparison with functions varying over scales larger than l^{-1}.

(ii) A second very important integral is

$$F_l(x) \equiv \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{\sin^2(k_x l)}{(k_x l)^2} \exp(ik_x x). \tag{2.30}$$

Note that this integral describes a function whose Fourier transform is presented by the squared first integral (2.29). After exact calculation we get

$$F_l(x) = \frac{1-|x|/2l}{2l}\Theta(2l-|x|). \tag{2.31}$$

In accordance with the explicit representation (2.31) the function $F_l(x)$ is different from zero only within the interval $(-2l < x < 2l)$ of width $4l$. It has a single maximum equal to $1/2l$ at the point $x = 0$ and monotonously decreases to the left and to the right of the maximum along the straight lines $2lF_l(x) = 1 \pm x/2l$ respectively. At the interval ends $x = \pm 2l$ the function $F_l(x)$ vanishes, $F_l(\pm 2l) = 0$. We underline that the integral of $F_l(x)$ along any symmetrical interval $(-2a, 2a)$ with $a > l$ is one:

$$\int_{-2a}^{2a} dx\, F_l(x) = \int_{-2a}^{2a} dx\, \frac{1-|x|/2l}{2l}\Theta(2l-|x|) = \int_{-2l}^{2l} \frac{dx}{2l}\left(1-\frac{|x|}{2l}\right) =$$

$$= 2\int_0^{2l} \frac{dx}{2l}\left(1-\frac{x}{2l}\right) = 1 \qquad \text{for} \qquad a > l. \tag{2.32}$$

Summarizing all the peculiarities of the function $F_l(x)$, we see that it is nothing else but the prelimit delta-function:

$$\lim_{l\to 0} F_l(x) = \delta(x). \tag{2.33}$$

Thus we can conclude that the function $F_l(x)$ is a sharp function and can be replaced by the delta-function within the variation scales much larger than the space averaging scale l. In particular, according to the conditions (2.19), we can do it within the macroscopic scales (either the wire length L or the mean free path L_{ts}) of variation of the smooth amplitudes $\pi_\pm(x)$ and $\gamma_\pm(x)$.

(iii) The third and last integral we should analyze is introduced by the expressions:

$$\Phi_l(x;k_F) \equiv \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \frac{\sin[(k_x \pm k_F)l]}{(k_x \pm k_F)l}\, \frac{\sin[(k_x \mp k_F)l]}{(k_x \mp k_F)l}\, \exp(ik_x x) =$$

$$= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \frac{\sin[(k_x - k_F)l]}{(k_x - k_F)l}\, \frac{\sin[(k_x + k_F)l]}{(k_x + k_F)l}\, \exp(ik_x x). \tag{2.34}$$

This integral describes a function whose Fourier transform is presented by the product of the two first integrals (2.29) with arguments shifted by $2k_F$ ($k_x \to k_x \mp k_F$ respectively). If k_F will be set to zero, the third integral (2.34) coincides with the second one (2.30):

$$\Phi_l(x;0) = F_l(x). \tag{2.35}$$

To evaluate the integral (2.34) we can use the residue theorem from the complex-variable theory. To this end we need to present the sines as sums of exponential functions and recall that k_F is in fact $k_F + i0$. The calculations are then performed in a way similar to that we applied to obtain the Fourier integral for the unperturbed Green function $\mathcal{G}_0(x-x';k_F)$ in the previous Lectures. So, we have

$$\Phi_l(x;k_F) = \frac{\sin[2k_F l(1-|x|/2l)]}{4k_F l^2}\,\Theta(2l-|x|). \tag{2.36}$$

We will now discuss the behavior of the function $\Phi_l(x;k_F)$ keeping in mind the relations (2.19) between different spatial scales. As well as $F_l(x)$, the function (2.36) is different from zero only within the interval $(-2l < x < 2l)$ of width $4l$. But in contrast to the former, $\Phi_l(x;k_F)$ rapidly oscillates within this interval. The scale of the oscillations is of the order of k_F^{-1} being much less than l ($k_F^{-1} \ll l$). Moreover, the amplitude of the oscillations, $1/4k_F l^2$, is $2k_F l \gg 1$ times less than the maximum $1/2l$ of the function $F_l(x)$. As a consequence of such oscillations the integral of $\Phi_l(x;k_F)$ along any symmetrical interval $(-2a, 2a)$ with $a > l$ is small with the parameter $(k_F l)^{-2} \ll 1$:

$$\int_{-2a}^{2a} dx\, \Phi_l(x;k_F) = \frac{\sin^2(k_F l)}{(k_F l)^2} \qquad \text{for} \qquad a > l. \tag{2.37}$$

Thus we come to the following conclusion: the function $\Phi_l(x;k_F)$ is small both locally and integrally. Therefore, within the same accuracy that allows to replace $F_l(x)$ by the delta-function the function $\Phi_l(x;k_F)$ can be set to zero,

$$\Phi_l(x;k_F) \approx 0. \tag{2.38}$$

Derivation of correlators for the random fields

After having discussed the useful integrals we are now in a position to calculate the binary correlators (2.24) of the random fields $\eta(x)$ and $\zeta_\pm(x)$.

(i) First we derive the simplest correlator $\langle\eta(x)\eta(x')\rangle$. By the definition (2.21),

$$\langle\eta(x)\eta(x')\rangle = \frac{1}{4k_F^2}\int_{x-l}^{x+l}\frac{dx_1}{2l}\int_{x'-l}^{x'+l}\frac{dx_2}{2l}\,\langle V(x_1)V(x_2)\rangle. \tag{2.39}$$

Use the notation (1.17) for the binary correlator of the random potential $V(x)$:

$$\langle\eta(x)\eta(x')\rangle = \frac{V_0^2}{4k_F^2}\int_{x-l}^{x+l}\frac{dx_1}{2l}\int_{x'-l}^{x'+l}\frac{dx_2}{2l}\,\mathcal{W}(|x_1-x_2|), \tag{2.40}$$

then substitute the Fourier representation (2.27) for the correlation coefficient $\mathcal{W}(|x_1-x_2|)$ into the integrand:

$$\langle\eta(x)\eta(x')\rangle = \frac{V_0^2}{4k_F^2}\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,W(k_x)\int_{x-l}^{x+l}\frac{dx_1}{2l}\,\exp(ik_xx_1)\int_{x'-l}^{x'+l}\frac{dx_2}{2l}\,\exp(-ik_xx_2). \tag{2.41}$$

Note that the integrals over x_1 and x_2 are defined explicitly by the first useful integral (2.29). So, we get

$$\langle\eta(x)\eta(x')\rangle = \frac{V_0^2}{4k_F^2}\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,W(k_x)\,\frac{\sin^2(k_xl)}{(k_xl)^2}\,\exp[ik_x(x-x')]. \tag{2.42}$$

The integrand of Eq. (2.42) contains two types of sharp functions. The first is the Fourier transform $W(k_x)$ with maximum at $k_x=0$ and variation scale $\Delta k_x \sim R_c^{-1}$. The second is $\sin^2(k_xl)/(k_xl)^2$ with the highest maximum at the same point $k_x=0$ but with variation scale $\Delta k_x \sim l^{-1}$. Owing to the relations (2.19), the second function is the sharpest ($l^{-1} \ll R_c^{-1}$) in the integrand. This fact allows to take the smoother factor $W(k_x)$ outside the integral at the point $k_x=0$:

$$\langle\eta(x)\eta(x')\rangle = \frac{V_0^2}{4k_F^2}\,W(0)\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,\frac{\sin^2(k_xl)}{(k_xl)^2}\,\exp[ik_x(x-x')]. \tag{2.43}$$

Let us apply the definition (2.25) for the forward-scattering length L_{fs} and find out that the integral over k_x is nothing else but the second useful integral (2.30). In this way we obtain the formula:

$$\langle\eta(x)\eta(x')\rangle = L_{fs}^{-1}\,F_l(x-x'). \tag{2.44}$$

Taking into account the preceding analysis, we replace the function $F_l(x-x')$ by the delta-function $\delta(x-x')$ and obtain the ultimate expression (2.24) for the binary correlator $\langle\eta(x)\eta(x')\rangle$ of the real random field $\eta(x)$.

(ii) By the definition (2.21) of the complex random fields $\zeta_{\pm}(x)$ we can write down the correlator $\langle \zeta_{\pm}(x)\zeta^{*}_{\pm}(x')\rangle$ in the form:

$$\langle \zeta_{\pm}(x)\zeta^{*}_{\pm}(x')\rangle \equiv \langle \zeta_{\pm}(x)\zeta_{\mp}(x')\rangle =$$

$$= \frac{1}{4k_F^2}\int_{x-l}^{x+l}\frac{dx_1}{2l}\exp(\pm 2ik_F x_1)\int_{x'-l}^{x'+l}\frac{dx_2}{2l}\exp(\mp 2ik_F x_2)\langle V(x_1)V(x_2)\rangle. \quad (2.45)$$

Use again the notation (1.17) for the binary correlator of the random potential $V(x)$ and substitute the Fourier representation (2.27) for the correlation coefficient $\mathcal{W}(|x_1 - x_2|)$ into the integrand:

$$\langle \zeta_{\pm}(x)\zeta^{*}_{\pm}(x')\rangle \equiv \langle \zeta_{\pm}(x)\zeta_{\mp}(x')\rangle = \frac{V_0^2}{4k_F^2}\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}W(k_x)\times$$

$$\times\int_{x-l}^{x+l}\frac{dx_1}{2l}\exp[i(k_x \pm 2k_F)x_1]\int_{x'-l}^{x'+l}\frac{dx_2}{2l}\exp[-i(k_x \pm 2k_F)x_2]. \quad (2.46)$$

The integrals over x_1 and x_2 are defined explicitly by the first useful integral (2.29). So, we have:

$$\langle \zeta_{\pm}(x)\zeta^{*}_{\pm}(x')\rangle \equiv \langle \zeta_{\pm}(x)\zeta_{\mp}(x')\rangle =$$

$$= \frac{V_0^2}{4k_F^2}\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}W(k_x)\frac{\sin^2[(k_x \pm 2k_F)l]}{[(k_x \pm 2k_F)l]^2}\exp[i(k_x \pm 2k_F)(x - x')]. \quad (2.47)$$

The integrand in Eq. (2.47) is similar to that in Eq. (2.42) for the correlator $\langle \eta(x)\eta(x')\rangle$. The only distinction is the shifted argument $k_x \pm 2k_F$ of the sharpest function $\sin^2[(k_x \pm 2k_F)l]/[(k_x \pm 2k_F)l]^2$. Therefore this function has now the highest maximum at the shifted pont $k_x = \mp 2k_F$. So, we should take the smoother coefficient $W(k_x)$ outside the integral at this point $k_x = \mp 2k_F$:

$$\langle \zeta_{\pm}(x)\zeta^{*}_{\pm}(x')\rangle \equiv \langle \zeta_{\pm}(x)\zeta_{\mp}(x')\rangle =$$

$$= \frac{V_0^2}{4k_F^2}W(\mp 2k_F)\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\frac{\sin^2[(k_x \pm 2k_F)l]}{[(k_x \pm 2k_F)l]^2}\exp[i(k_x \pm 2k_F)(x - x')]. \quad (2.48)$$

Now we take into account that the Fourier transform $W(k_x)$ of the correlation coefficient $W(x)$ is an even function of the argument k_x ($W(-k_x) = W(k_x)$) because $W(x)$ is even. In addition to this, we change the integration variable $k_x \to k_x^{new} = k_x \pm 2k_F$ and omit the index "*new*" in the final formula for simplicity. As a result, we get:

$$\langle \zeta_{\pm}(x)\zeta^{*}_{\pm}(x')\rangle \equiv \langle \zeta_{\pm}(x)\zeta_{\mp}(x')\rangle =$$

$$= \frac{V_0^2}{4k_F^2}W(2k_F)\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\frac{\sin^2(k_x l)}{(k_x l)^2}\exp[ik_x(x - x')]. \quad (2.49)$$

We note that by the definition (2.26) the factor before the integral is the inverse backscattering length L_{bs}^{-1}. At the same time the integral itself is nothing else but the second useful integral (2.30). Owing to this we obtain the formula:

$$\langle \zeta_{\pm}(x)\zeta_{\pm}^{*}(x')\rangle \equiv \langle \zeta_{\pm}(x)\zeta_{\mp}(x')\rangle = L_{bs}^{-1}\, F_l(x-x'). \tag{2.50}$$

Based on the previous analysis, we replace the function $F_l(x-x')$ by the delta-function $\delta(x-x')$ and get the ultimate expression (2.24) for the binary correlator $\langle \zeta_{\pm}(x)\zeta_{\pm}^{*}(x')\rangle$ of the complex random fields $\zeta_{\pm}(x)$.

(iii) Now we are going to calculate the cross-correlator $\langle \eta(x)\zeta_{\pm}(x')\rangle$. As before we use the explicit expressions (2.21) for $\eta(x)$ and $\zeta_{\pm}(x)$. After that we apply the formula (1.17) for the binary correlator of the scattering potential $V(x)$ and the Fourier representation (2.27) for the correlation coefficient $\mathcal{W}(|x_1-x_2|)$. As a result we obtain:

$$\begin{aligned}\langle \eta(x)\zeta_{\pm}(x')\rangle &= \frac{V_0^2}{4k_F^2}\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,W(k_x)\times \\ &\times \int_{x-l}^{x+l}\frac{dx_1}{2l}\,\exp(ik_x x_1)\int_{x'-l}^{x'+l}\frac{dx_2}{2l}\,\exp[-i(k_x\mp 2k_F)x_2].\end{aligned} \tag{2.51}$$

As usual, we next take the integrals over x_1 and x_2 with the aid of the first useful integral (2.29):

$$\begin{aligned}\langle \eta(x)\zeta_{\pm}(x')\rangle &= \frac{V_0^2}{4k_F^2}\,\exp(\pm 2ik_F x')\times \\ &\times \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,W(k_x)\,\frac{\sin(k_x l)}{k_x l}\,\frac{\sin[(k_x\mp 2k_F)l]}{(k_x\mp 2k_F)l}\,\exp[ik_x(x-x')].\end{aligned} \tag{2.52}$$

At this step of calculations we should discuss properties of the functions entering the integral over k_x in Eq. (2.52). Unlike the previous cases, it is crucial that the integrand of Eq. (2.52) now contains two (instead of one) equally sharp functions $\sin(k_x l)/k_x l$ and $\sin[(k_x\mp 2k_F)l]/(k_x\mp 2k_F)l$. They have the same variation scale $\Delta k_x \sim l^{-1}$ but different highest maxima at the points $k_x = 0$ and $k_x = \pm 2k_F$ respectively. Moreover, owing to the relations (2.19), the distance $2k_F$ between the maxima is much larger than their width l^{-1} ($l^{-1} \ll k_F$). All these facts require to take the smoother factor $W(k_x)$ outside the integral at the two mentioned points, $k_x = 0$ and $k_x = \pm 2k_F$:

$$\begin{aligned}\langle \eta(x)\zeta_{\pm}(x')\rangle &= \frac{V_0^2}{4k_F^2}\,[W(0)+W(\pm 2k_F)]\exp(\pm 2ik_F x')\times \\ &\times \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,\frac{\sin(k_x l)}{k_x l}\,\frac{\sin[(k_x\mp 2k_F)l]}{(k_x\mp 2k_F)l}\,\exp[ik_x(x-x')].\end{aligned} \tag{2.53}$$

We now make use of the definitions (2.25), (2.26) and (2.28) for the total relaxation length L_{ts} as well as of the evenness of the Fourier transform $W(k_x)$ ($W(-k_x) = W(k_x)$). Then we change the integration variable k_x for $k_x^{new} = k_x \mp k_F$ and omit the index "new" for simplicity:

$$\langle \eta(x)\zeta_{\pm}(x')\rangle = L_{ts}^{-1}\,\exp[\pm ik_F(x+x')]\,\times$$

$$\times \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,\frac{\sin[(k_x\pm k_F)l]}{(k_x\pm k_F)l}\,\frac{\sin[(k_x\mp k_F)l]}{(k_x\mp k_F)l}\,\exp[ik_x(x-x')]. \qquad (2.54)$$

The integral in the last expression (2.54) is nothing else but the third useful integral (2.34). So we obtain:

$$\langle \eta(x)\zeta_{\pm}(x')\rangle = L_{ts}^{-1}\,\exp[\pm ik_F(x+x')]\,\Phi_l(x-x';k_F). \qquad (2.55)$$

From the previous discussion (see the estimation (2.38)) we can conclude that the function $\Phi_l(x-x';k_F)$ and consequently the cross-correlator $\langle \eta(x)\zeta_{\pm}(x')\rangle$ vanish. This means that the ultimate expression (2.24) implies statistical independence of the random fields $\eta(x)$ and $\zeta_{\pm}(x)$.

(iv) The correlator $\langle \zeta_{\pm}(x)\zeta_{\pm}(x')\rangle$ of the random fields $\zeta_{\pm}(x)$ is derived similarly to the correlator $\langle \eta(x)\zeta_{\pm}(x')\rangle$. As always, we use the explicit expressions (2.21) for $\zeta_{\pm}(x)$. Then we apply the formula (1.17) for the binary correlator of the scattering potential $V(x)$ and the Fourier representation (2.27) for the correlation coefficient $\mathcal{W}(|x_1-x_2|)$. Finally we get:

$$\langle \zeta_{\pm}(x)\zeta_{\pm}(x')\rangle = \frac{V_0^2}{4k_F^2}\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,W(k_x)\,\times$$

$$\times \int_{x-l}^{x+l}\frac{dx_1}{2l}\,\exp[i(k_x\pm 2k_F)x_1]\int_{x'-l}^{x'+l}\frac{dx_2}{2l}\,\exp[-i(k_x\mp 2k_F)x_2]. \qquad (2.56)$$

We next take the integrals over x_1 and x_2 with the aid of the first useful integral (2.29):

$$\langle \zeta_{\pm}(x)\zeta_{\pm}(x')\rangle = \frac{V_0^2}{4k_F^2}\,\exp[\pm 2ik_F(x+x')]\,\times$$

$$\times \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,W(k_x)\,\frac{\sin[(k_x\pm 2k_F)l]}{(k_x\pm 2k_F)l}\,\frac{\sin[(k_x\mp 2k_F)l]}{(k_x\mp 2k_F)l}\,\exp[ik_x(x-x')]. \qquad (2.57)$$

The sharp functions in the integrand of Eq. (2.57) have maxima at the points $k_x=\mp 2k_F$ and $k_x=\pm 2k_F$. Therefore, we take the Fourier transform $W(k_x)$ of the correlation coefficient outside the integral at these two points:

$$\langle \zeta_{\pm}(x)\zeta_{\pm}(x')\rangle = \frac{V_0^2}{4k_F^2}\,[W(\mp 2k_F)+W(\pm 2k_F)]\,\exp[\pm 2ik_F(x+x')]\,\times$$

$$\times \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,\frac{\sin[(k_x\pm 2k_F)l]}{(k_x\pm 2k_F)l}\,\frac{\sin[(k_x\mp 2k_F)l]}{(k_x\mp 2k_F)l}\,\exp[ik_x(x-x')]. \qquad (2.58)$$

We now use the definition (2.26) for the backscattering length L_{bs} as well as the evenness of the Fourier transform $W(k_x)$ ($W(-k_x)=W(k_x)$) and the fact that the integral in the expression (2.58) is the third useful integral (2.34) with $2k_F$ instead of k_F. In this way we come to the result:

$$\langle\zeta_{\pm}(x)\zeta_{\pm}(x')\rangle = 2L_{bs}^{-1}\,\exp[\pm 2ik_F(x+x')]\,\Phi_l(x-x';2k_F). \tag{2.59}$$

According to the estimation (2.38) we can conclude that the function $\Phi_l(x-x';2k_F)$ and consequently the correlator $\langle\zeta_{\pm}(x)\zeta_{\pm}(x')\rangle$ vanish. It is exactly what we have written for the correlator of the random fields $\zeta_{\pm}(x)$ in the ultimate expressions (2.24).

2.2.3 Symbolic solution of equations for smooth amplitudes. Unimodularity relations

Our four first-order differential equations (2.20) are linear equations with four initial conditions (2.11). The coefficients $\eta(x)$ and $\zeta_{\pm}(x)$ of the equations are continuous functions within the variation interval of the variable x. Therefore the theory of differential equations states that a solution of the equations exists and it is unique. Unfortunately, it is highly unlikely to obtain this solution in a finite analytic form because of the complicated dependencies (2.21) of the variable coefficients $\eta(x)$ and $\zeta_{\pm}(x)$ on the "time" x. Nevertheless, we can write down the solution of the set (2.20) and (2.11) in some symbolic matrix form which allows us to derive the very important so-called *unimodularity relations* for the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$.

Let us introduce matrices $I_{\pm}(x)$ of the unknown smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$:

$$I_+(x) = \begin{pmatrix} \pi_+(x) & \gamma_+(x) \\ \gamma_+^*(x) & \pi_+^*(x) \end{pmatrix}, \qquad I_-(x) = \begin{pmatrix} \pi_-(x) & \gamma_-^*(x) \\ \gamma_-(x) & \pi_-^*(x) \end{pmatrix}. \tag{2.60}$$

Then we define the matrix $\hat{b}(x)$ of the random fields $\eta(x)$ and $\zeta_{\pm}(x)$ as

$$\hat{b}(x) = \begin{pmatrix} i\eta(x) & \zeta_+(x) \\ \zeta_-(x) & -i\eta(x) \end{pmatrix}. \tag{2.61}$$

We draw attention to the fact that the matrix $\hat{b}(x)$ is traceless:

$$\mathrm{Tr}\,\hat{b}(x) \equiv i\eta(x) - i\eta(x) = 0. \tag{2.62}$$

In the new matrix notation (2.60) and (2.61) the dynamic equations (2.20) take the following matrix representations:

$$I'_+(x) + I_+(x)\hat{b}(x) = 0, \tag{2.63}$$

$$I'_-(x) - \hat{b}(x)I_-(x) = 0. \tag{2.64}$$

In addition, due to initial conditions (2.11) the matrices $I_{\pm}(x)$ of the smooth amplitudes are unit matrices at the points $x = \pm L/2$ respectively:

$$I_{\pm}(\pm L/2) = \hat{\mathbf{1}} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.65}$$

From the evolutional equations (2.63) and (2.64) it follows that the first derivatives $I'_{\pm}(x)$ of the functions $I_{\pm}(x)$ are equal to the functions themselves multiplied by the variable factors $\mp\hat{b}(x)$. We know that solutions of such equations are exponential functions with the variable factors $\mp\hat{b}(x)$ being the derivatives of the exponents. So, regardless of

the matrix nature of the equations (2.63) and (2.64), we can state that their solutions satisfying the initial conditions (2.65) are given by the following exponential functions:

$$\exp\left[\pm\int_{x}^{\pm L/2} dx'\,\hat{b}(x')\right]. \tag{2.66}$$

We stress that the matrix $I_{+}(x)$ is multiplied by the matrix $\hat{b}(x)$ from the right in the equation (2.63) while the matrix $I_{-}(x)$ is multiplied by the matrix $\hat{b}(x)$ from the left in the equation (2.64). This distinction is very important because in general the commutative law does not hold for matrix multiplication ($\hat{a}\hat{b} \neq \hat{b}\hat{a}$). We then note that in Eq. (2.66) the expression with the "+" sign, which is related to the matrix $I_{+}(x)$, contains the matrix $\hat{b}(x')$ whose argument x' is greater than x ($-L/2 \leq x \leq x' \leq L/2$) while the expression with the "$-$" sign, which is associated with the matrix $I_{-}(x)$, contains the matrix $\hat{b}(x')$ whose argument x' is smaller than x ($-L/2 \leq x' \leq x \leq L/2$). Summarizing, we see that in both equations (2.63) and (2.64) the matrix multiplications are such that the matrix $\hat{b}$ with a larger argument stands to the left of the matrix $\hat{b}$ with a smaller argument. Therefore, to account for this fact properly we need to introduce the ordering operator $\hat{T}_x$ in the coordinate x. This *x-ordering operator $\hat{T}_x$ arranges all matrices $\hat{b}$ as well as their functions to its right so that any matrix with a larger argument is placed to the left of any matrix with a smaller argument.* With the use of the ordering operator we can write the required solutions as

$$I_{\pm}(x) = \hat{T}_x \exp\left[\pm\int_{x}^{\pm L/2} dx'\,\hat{b}(x')\right]. \tag{2.67}$$

By a straightforward verification we can make sure that the structure (2.67) of the matrices $I_{\pm}(x)$ of the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$ are really the solutions of the equations (2.63) and (2.64), respectively.

Let us apply the well-known identity for determinant of a matrix:

$$\ln\det\hat{A} \equiv \mathrm{Tr}\,\ln\hat{A}, \qquad \text{i.e.} \qquad \det\hat{A} \equiv \exp(\mathrm{Tr}\,\ln\hat{A}). \tag{2.68}$$

In accordance with this property we easily find that the matrices $I_{\pm}(x)$ are *unimodular*, i.e. their determinants are equal to one:

$$\det I_{\pm}(x) = \exp\left[\mathrm{Tr}\,\ln I_{\pm}(x)\right] = \exp\left[\pm\mathrm{Tr}\int_{x}^{\pm L/2} dx'\hat{b}(x')\right] = 1. \tag{2.69}$$

This conclusion is also originated from the tracelessness (2.62) of the matrix $\hat{b}(x)$. On the other hand, by the definition (2.60) the determinants of the matrices $I_{\pm}(x)$ are expressed via the difference between squared moduli of the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$:

$$\det I_{\pm}(x) = |\pi_{\pm}(x)|^2 - |\gamma_{\pm}(x)|^2. \tag{2.70}$$

Thus, we arrive at an essential statement: *the structure of the dynamic equations (2.20) is such that the the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$ introduced within the two-scale model (2.5) turn out to be related by the unimodularity conditions*:

$$|\pi_{\pm}(x)|^2 - |\gamma_{\pm}(x)|^2 = 1. \tag{2.71}$$

2.3 Transmittance and Reflectance of 1D Disordered Conductor

2.3.1 Landauer formula for conductance

A next step of our investigation within the two-scale description (2.5) is to express the dimensionless conductance (2.2) via the smooth amplitudes $\pi_\pm(x)$ and $\gamma_\pm(x)$. To do this we substitute the single-electron Green functions in the form (2.6) into the Kubo formula (2.3):

$$\begin{aligned} T(L) &= -\frac{4}{L^2}\frac{1}{|W|^2}\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \times \\ &\times \left[\psi'_+(x)\psi_-(x')\Theta(x-x') + \psi_+(x')\psi'_-(x)\Theta(x'-x)\right] \times \\ &\times \left[\psi^*_+(x)\psi^{*\prime}_-(x')\Theta(x-x') + \psi^{*\prime}_+(x')\psi^*_-(x)\Theta(x'-x)\right] = \\ &= -\frac{1}{|W|^2}\frac{4}{L^2}\int_{-L/2}^{L/2} dx \left[\psi'_+(x)\psi^*_+(x)\int_{-L/2}^{x} dx'\psi_-(x')\psi^{*\prime}_-(x') + \right. \\ &+ \left. \psi'_-(x)\psi^*_-(x)\int_{x}^{L/2} dx'\psi_+(x')\psi^{*\prime}_+(x')\right]. \end{aligned} \tag{2.72}$$

We see that various products of the wave functions $\psi_\pm(x)$ by their derivatives appear in the equation (2.72), viz $\psi'_\pm(x)\psi^*_\pm(x)$ and $\psi_\pm(x)\psi^{*\prime}_\pm(x)$. To perform operations in these products we should take into account the experience of the previous calculations within the two-scale model (2.5): First, we use the two-scale representation (2.9) for the wave functions $\psi_\pm(x)$. Second, based on the exact expression (2.13) for the first derivatives of $\psi_\pm(x)$ and the estimations (2.15) we should differentiate the rapidly oscillating exponents $\exp(\pm ik_F x)$ and $\exp(\mp ik_F x)$ only when deriving the functions $\psi'_\pm(x)$. Third, after multiplying out the wave functions $\psi_\pm(x)$ and their derivatives we should neglect terms containing the rapidly oscillating exponents and retain terms in which the rapidly oscillating exponents cancel out. As a result we get

$$\begin{aligned} \psi'_\pm(x)\psi^*_\pm(x) &\approx \pm ik_F\left[|\pi_\pm(x)|^2 - |\gamma_\pm(x)|^2\right] = \pm ik_F, \\ \psi_\pm(x)\psi^{*\prime}_\pm(x) &\approx \mp ik_F\left[|\pi_\pm(x)|^2 - |\gamma_\pm(x)|^2\right] = \mp ik_F. \end{aligned} \tag{2.73}$$

We draw attention to the following surprising fact. In accordance with the unimodularity relations (2.71), the products (2.73) and consequently the integrand in the equation (2.72) turn out to be independent of the smooth amplitudes $\pi_\pm(x)$ and $\gamma_\pm(x)$ and hence of the integration variables x and x'. Thus we come to the conclusion that the conductance is determined only by the squared modulus of the Wronskian W of the wave functions $\psi_\pm(x)$:

$$T(L) = 4k_F^2|W|^{-2}. \tag{2.74}$$

We now need to obtain the Wronskian (2.7) within the two-scale approximation (2.5). As before, substituting the wave functions $\psi_{\pm}(x)$ (2.9) in Eq. (2.7) we should differentiate the rapidly oscillating exponents only and then multiplying the wave functions by the derivatives we should retain only such terms in which the rapidly oscillating exponents cancel out. In this way we have

$$W = 2ik_F \left[\pi_+(x)\pi_-(x) + \gamma_+(x)\gamma_-(x)\right] = 2ik_F \pi_{\pm}(\mp L/2). \tag{2.75}$$

Here, in the last equality of Eq. (2.75), we have taken into account that the Wronskian W of any Hermitian problem does not depend on the coordinate x and we can calculate it at any convenient point within the interval (2.1). In accordance with the initial conditions (2.11) for the smooth amplitudes ($\pi_{\pm}(\pm L/2) = 1, \gamma_{\pm}(\pm L/2) = 0$), the most convenient points are the wire ends $x = \pm L/2$.

Substitution of the expression (2.75) into the formula (2.74) gives us the required relation between the dimensionless conductance (2.2) and the smooth amplitudes $\pi_{\pm}(x)$,

$$T(L) = |\pi_{\pm}^{-1}(\mp L/2)|^2. \tag{2.76}$$

Following the Landauer idea [15, 16] we assume that the dimensionless conductance of a 1D disordered conductor is equal to the squared modulus of the amplitude transmission coefficient. From this assumption and the equality (2.76) we can figure out that the quantity $|\pi_{\pm}^{-1}(\mp L/2)|$ can be regarded as the modulus of the amplitude transmission coefficient of a conducting 1D disordered wire of the length L.

2.3.2 Transmission and reflection coefficients

A widely accepted and physically clear approach to studying transport properties and in particular the conductance of one-dimensional and quasi-one-dimensional systems is to express the conductance via the transmission and reflection coefficients of electron waves scattered by the inhomogeneous structure. By analogy with the scattering theory we introduce *the transmission* $\Upsilon_{\pm}(x)$ *and reflection* $\Gamma_{\pm}(x)$ *local amplitude coefficients for an electron wave.*

The equality (2.76) for the conductance makes it clear that the modulus of the local transmission coefficient $\Upsilon_{\pm}(x)$ must be equal to

$$|\Upsilon_{\pm}(x)| = |\pi_{\pm}^{-1}(x)|. \tag{2.77}$$

At the same time the transmission and reflection coefficients must obey *the flow conservation law*:

$$|\Upsilon_{\pm}(x)|^2 + |\Gamma_{\pm}(x)|^2 = 1. \tag{2.78}$$

From the unimodularity relations (2.71) for the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$ it can be easily established that the modulus of the amplitude reflection coefficient $\Gamma_{\pm}(x)$ satisfies the flow conservation law (2.78) as long as

$$|\Gamma_{\pm}(x)| = |\gamma_{\pm}(x)/\pi_{\pm}(x)|. \tag{2.79}$$

We see that only the moduli of the transmission and reflection coefficients are defined uniquely from physical reasoning, while their phases can be fixed by additional considerations.

The structure of the dynamic equations (2.20) is such that the real random field $\eta(x)$ determines only the phases of the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$. In other words the forward-scattering field $\eta(x)$ can be eliminated from these equations by concurrent phase transformations of the smooth amplitudes and the complex random fields $\zeta_{\pm}(x)$. In view of this fact let us introduce the transmission $\Upsilon_{\pm}(x)$ and reflection $\Gamma_{\pm}(x)$ amplitude local coefficients according to the following two conditions: (i) Their moduli coincide with Eqs. (2.77) and (2.79), respectively. (ii) The equations for them do not contain the random field $\eta(x)$ in explicit form. All these conditions are satisfied by the expressions:

$$\Upsilon_{\pm}(x) = \frac{1}{\pi_{\pm}(x)} \exp\left[\mp i \int_{\pm L/2}^{x} dx' \eta(x')\right]. \tag{2.80}$$

$$\Gamma_{\pm}(x) = \frac{\gamma_{\pm}(x)}{\pi_{\pm}(x)} \exp\left[\mp 2i \int_{\pm L/2}^{x} dx' \eta(x')\right]. \tag{2.81}$$

Indeed, the moduli of Eqs. (2.80) and (2.81) are given by the formulas (2.77) and (2.79), respectively. *The equations for the transmission and reflection coefficients* are obtained by a straightforward derivation of the definitions (2.80) and (2.81). Using the dynamic equations (2.20) for the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$ we get

$$\pm \frac{d\Upsilon_{\pm}(x)}{dx} = \zeta_{\pm}^{new}(x)\Gamma_{\pm}(x)\Upsilon_{\pm}(x), \tag{2.82}$$

$$\pm \frac{d\Gamma_{\pm}(x)}{dx} = \zeta_{\pm}^{new}(x)\Gamma_{\pm}^{2}(x) - \zeta_{\pm}^{new*}(x). \tag{2.83}$$

So, the real random field $\eta(x)$ certainly does not enter these equations explicitly.

According to the initial conditions (2.11) for the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$, the equations (2.82) and (2.83) are complemented by the following *initial conditions*:

$$\Upsilon_{\pm}(\pm L/2) = 1, \qquad \Gamma_{\pm}(\pm L/2) = 0. \tag{2.84}$$

Note, the lower integration limits $\pm L/2$ in the expressions (2.80) and (2.81) have been taken such that the exponential factors do not affect the initial conditions (2.84) for the transmission and reflection local coefficients.

The new variable coefficients $\zeta_{\pm}^{new}(x)$ appear in the equations (2.82) and (2.83). They are related to the old complex random fields $\zeta_{\pm}(x)$ by

$$\zeta_{\pm}^{new}(x) = \zeta_{\mp}(x) \exp\left[\pm 2i \int_{\pm L/2}^{x} dx' \eta(x')\right]. \tag{2.85}$$

As well as $\zeta_{\pm}(x)$, the new renormalized random fields $\zeta_{\pm}^{new}(x)$ are complex conjugate to each other,

$$\zeta_{\pm}^{new*}(x) = \zeta_{\mp}^{new}(x). \tag{2.86}$$

Now we would like to make some significant remarks.

The introduced transmission $\Upsilon_{\pm}(x)$ and reflection $\Gamma_{\pm}(x)$ amplitude local coefficients have the following physical meaning: The coefficient $\Upsilon_{+}(x)$ is the amplitude coefficient of transmission of an electron wave through the interval $(x, L/2)$ when impinged the interval from the left. The coefficient $\Gamma_{+}(x)$ is the amplitude coefficient of reflection of the same electron wave and for the same interval $(x, L/2)$. Regardless of the quantities with the symbol "+", the coefficient $\Upsilon_{-}(x)$ is the amplitude coefficient of transmission of an electron wave through the other interval $(-L/2, x)$ when impinged the interval from the right. The coefficient $\Gamma_{-}(x)$ is the amplitude coefficient of reflection of the same electron wave from the interval $(-L/2, x)$. One can clearly understand that the first of the mentioned electron waves is described by the formula with the factor $\exp(ik_F x)$ while the second is described by the expression proportional to $\exp(-ik_F x)$.

All the equations obtained, i.e. Eq. (2.20) for the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$ as well as Eqs. (2.82) and (2.83) for the transmission $\Upsilon_{\pm}(x)$ and reflection $\Gamma_{\pm}(x)$ coefficients, together with their initial conditions (2.11) and (2.84) respectively, are *the dynamic evolutional equations*. Therefore their solutions are presented by *causal functionals of the random fields* $\eta(x)$ and $\zeta_{\pm}(x)$, or $\zeta_{\pm}^{new}(x)$. This means that all the solutions with the index "+" at a given point x are specified by values of the random fields only within the interval $(x, L/2)$, whereas all the solutions with the index "−" at the same point x are determined by values of the random fields only within the other interval $(-L/2, x)$. For the smooth amplitudes $\pi_{\pm}(x)$ and $\gamma_{\pm}(x)$ these statements result from the symbolic solutions (2.67). In general, we can easily verify the validity of the previous statements by a direct integration of the corresponding dynamic differential equations with the use of the corresponding initial conditions.

In particular, *the reflection coefficient* $\Gamma_{\pm}(x)$ *satisfies the Riccati-type nonlinear evolutional equation* (2.83) *with the homogeneous initial conditions* (2.84). It can be easily shown that the set (2.83) and (2.84) is equivalent to the following integral equation:

$$\Gamma_{\pm}(x) = \pm \int_{\pm L/2}^{x} dx' \left[\zeta_{\pm}^{new}(x')\Gamma_{\pm}^{2}(x') - \zeta_{\pm}^{new*}(x')\right]. \tag{2.87}$$

From this integral equation we see that the reflection amplitude local coefficients $\Gamma_{\pm}(x)$ actually belong to causal functionals of the renormalized complex random fields $\zeta_{\pm}^{new}(x)$.

The above remark is essential for the averaging procedure because, owing to the delta-correlation of the random fields, it allows to average products of any functionals with different indexes and arguments independently.

2.3.3 Correlators of new random fields

For further calculations we need to know the statistical properties of the complex random fields $\zeta_{\pm}^{new}(x)$.

To start with let us derive the averages of these fields. From the definition (2.85) we have

$$\langle \zeta_{\pm}^{new}(x) \rangle = \langle \zeta_{\mp}(x) \exp\left[\pm 2i \int_{\pm L/2}^{x} dx' \eta(x')\right] \rangle. \tag{2.88}$$

In consequence of the statistical independence (2.24) of the old fields $\zeta_{\pm}(x)$ and $\eta(x)$, the average of the product on the r.h.s. of the equation (2.88) is equal to the product of the averaged multipliers:

$$\langle\zeta_{\pm}^{new}(x)\rangle = \langle\zeta_{\mp}(x)\rangle\langle\exp\left[\pm 2i\int_{\pm L/2}^{x} dx'\eta(x')\right]\rangle. \tag{2.89}$$

Let us recall that the old random fields $\zeta_{\pm}(x)$ have zero averages (2.23). Owing to this, we obtain that the averages of the renormalized fields $\zeta_{\pm}^{new}(x)$ are equal to zero too:

$$\langle\zeta_{\pm}^{new}(x)\rangle = 0. \tag{2.90}$$

In the same way we come to the zero result for the binary correlator:

$$\begin{aligned}\langle\zeta_{\pm}^{new}(x)\zeta_{\pm}^{new}(x')\rangle &= \langle\zeta_{\mp}(x)\zeta_{\mp}(x')\exp\left[\pm 2i\left(\int_{\pm L/2}^{x}+\int_{\pm L/2}^{x'}\right)dx_1\eta(x_1)\right]\rangle = \\ &= \langle\zeta_{\mp}(x)\zeta_{\mp}(x')\rangle\langle\exp\left[\pm 2i\left(\int_{\pm L/2}^{x}+\int_{\pm L/2}^{x'}\right)dx_1\eta(x_1)\right]\rangle = \\ &= 0. \end{aligned}\tag{2.91}$$

Here we have exploited the fact that according to Eq. (2.24) the binary correlator of the old random fields $\zeta_{\pm}(x)$ vanishes ($\langle\zeta_{\mp}(x)\zeta_{\mp}(x')\rangle = 0$).

Finally we should obtain the last correlator $\langle\zeta_{\pm}^{new}(x)\zeta_{\pm}^{new*}(x')\rangle$. By the definition (2.85) this correlator is

$$\begin{aligned}\langle\zeta_{\pm}^{new}(x)\zeta_{\pm}^{new*}(x')\rangle &= \langle\zeta_{\mp}(x)\zeta_{\mp}^{*}(x')\exp\left[\pm 2i\left(\int_{\pm L/2}^{x}-\int_{\pm L/2}^{x'}\right)dx_1\eta(x_1)\right]\rangle = \\ &= \langle\zeta_{\mp}(x)\zeta_{\mp}^{*}(x')\rangle\langle\exp\left[\pm 2i\int_{x'}^{x}dx_1\eta(x_1)\right]\rangle. \end{aligned}\tag{2.92}$$

We substitute here the expression for the correlator $\langle\zeta_{\mp}(x)\zeta_{\mp}^{*}(x')\rangle$ of the old complex fields $\zeta_{\pm}(x)$ from Eq. (2.24). Then we take into account that, because of the delta-function $\delta(x-x')$, the exponent of the exponential function in Eq. (2.92) vanishes. In this way we arrive at the result:

$$\langle\zeta_{\pm}^{new}(x)\zeta_{\pm}^{new*}(x')\rangle = L_{bs}^{-1}\delta(x-x'). \tag{2.93}$$

Thus we see that the new renormalized fields $\zeta_{\pm}^{new}(x)$ possess the same correlations properties as the old random fields $\zeta_{\pm}(x)$ do. Hence, as well as the old fields, we can consider the new fields $\zeta_{\pm}^{new}(x)$ *a complex Gaussian "white noise"*.

2.3.4 Intermediate summary

According to equality (2.76) and definition (2.80) *the dimensionless conductance* (2.2) *is nothing else but the squared modulus of the amplitude coefficient* $\Upsilon_{\pm}(\mp L/2)$ *of electron transmission trough all 1D disordered wire of length* L. Therefore, to solve the conductance problem we need to find only the transmission local amplitude coefficient $\Upsilon_{\pm}(x)$ and then take its squared modulus at the points $x = \mp L/2$. For this purpose we would want to solve the equation (2.82) with the initial condition (2.84).

Unfortunately, the equation (2.82) for the transmission coefficient $\Upsilon_{\pm}(x)$ is not closed. Indeed, to solve it we must know $\Gamma_{\pm}(x)$. At the same time, we see that the equation (2.83) for the reflection coefficient $\Gamma_{\pm}(x)$ is closed and, in addition, complemented by the homogeneous initial condition (2.84). Hence, *it is more convenient to deal with the reflection local amplitude coefficient* $\Gamma_{\pm}(x)$ rather than the transmission coefficient $\Upsilon_{\pm}(x)$.

Within this approach the dimensionless conductance $T(L)$ is expressed via the squared modulus of the reflection coefficient $\Gamma_{\pm}(x)$ with the use of the flow conservation law (2.78),

$$T(L) = 1 - |\Gamma_{\pm}(\mp L/2)|^2. \tag{2.94}$$

As usual, let us call *the squared modulus* $|\Upsilon_{\pm}(x)|^2$ *of the transmission coefficient* $\Upsilon_{\pm}(x)$ *the local transmittance* while *the squared modulus* $|\Gamma_{\pm}(x)|^2$ *of the reflection coefficient* $\Gamma_{\pm}(x)$ will be called *the local reflectance*. So, we will perform all forthcoming calculations in terms of the reflectance.

We did emphasize above that the reflection coefficient $\Gamma_{\pm}(x)$ is in fact a causal functional of the renormalized complex random fields $\zeta_{\pm}^{new}(x)$. Obviously, for the averaging procedure the explicit definition (2.85) of these fields does not matter. Only the correlation relations (2.90), (2.91) and (2.93) are important. At the same time, the non-zero correlator (2.93) is specified by the electron backscattering length L_{bs} and does not depend on the forward scattering mean free path L_{fs}. Thus, we come to a non-trivial conclusion: *averaged values of the reflectance and conductance are only determined by the backscattering of electrons while forward scattering is removed.*

Since the new renormalized fields $\zeta_{\pm}^{new}(x)$ possess the same correlations properties as the old random fields $\zeta_{\pm}(x)$ we will omit the symbol "*new*" in all further calculations.

To treat the reflectance problem we are going to state below a very powerful and useful method which allows to calculate not only the averaged reflectance, conductance and resistance but any of their moments as well.

2.4 Moments of Reflectance

The averaged value of an n-th power of a random quantity is often called an n-th moment of this quantity. Now let us introduce *the n-th moment*, $R_n^{\pm}(x)$, *of the local reflectance* $|\Gamma_{\pm}(x)|^2$,

$$R_n^{\pm}(x) \equiv \langle |\Gamma_{\pm}(x)|^{2n} \rangle = \langle \Big[\Gamma_{\pm}(x) \Gamma_{\pm}^*(x) \Big]^n \rangle, \qquad n = 0, 1, 2, 3 \ldots \ . \tag{2.95}$$

It is necessary to underline that the moment (2.95) represents the average of functions taken at a same coordinate. If the conditions of statistical homogeneity are satisfied, then such averages do not depend on the coordinate. At the same time, in line with our above assumptions, we need to assume that the moment $R_n^{\pm}(x)$ depends on the coordinate x and varies over either the backscattering length L_{bs} or the wire length L. This fact leads to the statistical inhomogeneity of our problem over macroscopic scales of the order of both L_{bs} and L. We note that within the two-scale approach (2.5) quantities varying over microscopic scales (for example, over the correlation radius R_c) can be considered statistical homogeneous. This is exactly the reason why the dispersion $\langle V^2(x) \rangle = V_0^2$ of the random potential $V(x)$ does not depend on the coordinate x and the correlation coefficient $\mathcal{W}(|x - x'|)$ depends only on the difference $x - x'$.

So, if we derive the n-th reflectance moment $R_n^{\pm}(x)$, then, according to the equation (2.94), the average dimensionless conductance $\langle T(L)\rangle$ is given by the first moment $R_1^{\pm}(x)$ taken at the point $x = \mp L/2$,

$$\langle T(L)\rangle = 1 - R_1^{\pm}(\mp L/2). \tag{2.96}$$

One way to find $R_n^{\pm}(x)$ is to deduce a corresponding equation with initial conditions and then solve it.

2.4.1 Differential-difference equation for reflectance moment

Let us differentiate the definition (2.95) with the use of the equation (2.83) for $\Gamma_{\pm}(x)$ and the corresponding equation for $\Gamma_{\pm}^*(x)$,

$$\pm\frac{dR_n^{\pm}(x)}{dx} = 2n\Re\left[\langle\Gamma_{\pm}^{n+1}(x)\Gamma_{\pm}^{*n}(x)\zeta_{\pm}(x)\rangle - \langle\Gamma_{\pm}^{n}(x)\Gamma_{\pm}^{*(n-1)}(x)\zeta_{\pm}(x)\rangle\right]. \tag{2.97}$$

Please, recall that the symbol "$\Re$" stands for the real part.

Every term (correlator) on the r.h.s. of the equation (2.97) is obtained with the aid of the Furutsu-Novikov formula:

$$\langle\zeta(x)\Phi(x)\rangle = \int dx'\langle\zeta(x)\zeta^*(x')\rangle\langle\frac{\delta\Phi(x)}{\delta\zeta^*(x')}\rangle. \tag{2.98}$$

Here $\zeta(x)$ is a complex Gaussian random process (field) with zero average ($\langle\zeta(x)\rangle = 0$), and $\Phi(x)$ is a functional of this field. In calculating variational derivatives one must consider the fields $\zeta(x)$ and $\zeta^*(x)$ as independent variables, i.e.

$$\frac{\delta\zeta(x)}{\delta\zeta(x')} = \frac{\delta\zeta^*(x)}{\delta\zeta^*(x')} = \delta(x-x'), \qquad \frac{\delta\zeta(x)}{\delta\zeta^*(x')} = \frac{\delta\zeta^*(x)}{\delta\zeta(x')} = 0. \tag{2.99}$$

In the Furutsu-Novikov formula (2.98) the integral over x' is taken over the definition interval of the random field $\zeta(x)$.

Applying the Furutsu-Novikov formula (2.98) to the first term on the r.h.s. of the equation (2.97) we have

$$\langle\Gamma_{\pm}^{n+1}(x)\Gamma_{\pm}^{*n}(x)\zeta_{\pm}(x)\rangle = \int_{-L/2}^{L/2} dx'\langle\zeta_{\pm}(x)\zeta_{\pm}^*(x')\rangle\langle\frac{\delta\Gamma_{\pm}^{n+1}(x)\Gamma_{\pm}^{*n}(x)}{\delta\zeta_{\pm}^*(x')}\rangle. \tag{2.100}$$

So, we need to calculate the variational derivative in the integrand of Eq. (2.100). We note that the variational derivative of a complicated functional is calculated according to the same rules as the ordinary derivative,

$$\frac{\delta\Gamma_{\pm}^{n+1}(x)\Gamma_{\pm}^{*n}(x)}{\delta\zeta_{\pm}^*(x')} = (n+1)|\Gamma_{\pm}(x)|^{2n}\frac{\delta\Gamma_{\pm}(x)}{\delta\zeta_{\pm}^*(x')} + n\Gamma_{\pm}^{n+1}(x)\Gamma_{\pm}^{*(n-1)}(x)\frac{\delta\Gamma_{\pm}^*(x)}{\delta\zeta_{\pm}^*(x')}. \tag{2.101}$$

The variational derivatives of the reflection coefficient $\Gamma_{\pm}(x)$ and its complex conjugate $\Gamma_{\pm}^*(x)$ are obtained by differentiating the integral equation (2.87) and the corresponding complex conjugate equation, respectively,

$$\frac{\delta\Gamma_{\pm}(x)}{\delta\zeta_{\pm}^{*}(x')} = \Theta(\pm[x'-x])\left[1 \mp 2\int_{x}^{x'} dx''\zeta_{\pm}(x'')\Gamma_{\pm}(x'')\frac{\delta\Gamma_{\pm}(x'')}{\delta\zeta_{\pm}^{*}(x')}\right];$$
$$\frac{\delta\Gamma_{\pm}^{*}(x)}{\delta\zeta_{\pm}^{*}(x')} = -\Theta(\pm[x'-x])\left[\Gamma_{\pm}^{*2}(x') \pm 2\int_{x}^{x'} dx''\zeta_{\pm}^{*}(x'')\Gamma_{\pm}^{*}(x'')\frac{\delta\Gamma_{\pm}^{*}(x'')}{\delta\zeta_{\pm}^{*}(x')}\right]. \tag{2.102}$$

We should remember that the variational derivatives of a causal integral functional of some field with respect to this field differ from zero only if the argument of the field falls into the integration interval. Taking into account the integration path in the integral equation (2.87), we conclude that $\delta\Gamma_{\pm}(x)/\delta\zeta_{\pm}^{*}(x')$ and $\delta\Gamma_{\pm}^{*}(x)/\delta\zeta_{\pm}^{*}(x')$ are proportional to the Θ-functions $\Theta(\pm[x'-x])$. Consequently, the variational derivatives standing in the integrals over x'' are proportional to $\Theta(\pm[x'-x''])$. Therefore in the expressions (2.102) we have written down explicitly the first mentioned Θ-functions and cut off the limits of the integration over x'' due to the second Θ-functions.

Now let us substitute the expressions (2.102) into Eq. (2.101). Then we substitute the result into the Furutsu-Novikov formula (2.100) and cut off the integration limits over x' with the use of $\Theta(\pm[x'-x])$. After that we apply the explicit form (2.93) for the binary correlator $\langle\zeta_{\pm}(x)\zeta_{\pm}^{*}(x')\rangle$. Owing to the delta-correlation of the random field $\zeta_{\pm}(x)$ the integral over x' can be taken and gives a finite answer whereas the integrals over x'' vanish. So, taking into account the definition (2.95) we get the following result for the first term on the r.h.s. of the equation (2.97):

$$\langle\Gamma_{\pm}^{n+1}(x)\Gamma_{\pm}^{*n}(x)\zeta_{\pm}(x)\rangle = \frac{1}{2L_{bs}}\left[(n+1)R_{n}^{\pm}(x) - nR_{n+1}^{\pm}(x)\right]. \tag{2.103}$$

Analogous calculations of the second term on the r.h.s. of the equation (2.97) yield

$$\langle\Gamma_{\pm}^{n}(x)\Gamma_{\pm}^{*(n-1)}(x)\zeta_{\pm}(x)\rangle = \frac{1}{2L_{bs}}\left[nR_{n-1}^{\pm}(x) - (n-1)R_{n}^{\pm}(x)\right]. \tag{2.104}$$

Substituting the expressions (2.103) and (2.104) into the starting equation (2.97) we arrive at the ultimate form of *the differential-difference equation for the reflectance moment* $R_{n}^{\pm}(x)$,

$$\pm\frac{dR_{n}^{\pm}(x)}{dx} = -\frac{n^2}{L_{bs}}\left[R_{n+1}^{\pm}(x) - 2R_{n}^{\pm}(x) + R_{n-1}^{\pm}(x)\right], \qquad n = 0, 1, 2, 3\ldots. \tag{2.105}$$

We need to discuss now the initial conditions for the equation (2.105). We point out that the reflectance moment $R_{n}^{\pm}(x)$ is a function of two variables: the coordinate x and the index n. Hence, we should formulate initial conditions for every variable, both x and n.

It is necessary to underline that the difference equation (2.105) relates three reflectance moments $R_{n+1}^{\pm}(x)$, $R_{n}^{\pm}(x)$ and $R_{n-1}^{\pm}(x)$. This means that it is equivalent to the second order differential equation with respect to the index n and therefore requires *two initial conditions on* n. They follow naturally from the definition (2.95). Actually, when the index n is equal to zero, the moment $R_{n}^{\pm}(x)$ equals one. Next, according to the flow conservation law (2.78), the reflectance $|\Gamma_{\pm}(x)|^2$ is always smaller than one ($|\Gamma_{\pm}(x)|^2 \leq$

1). Consequently, the moment $R_n^{\pm}(x)$ tends to zero while the index n tends to infinity ($R_n^{\pm}(x) \to 0$ as $n \to \infty$). So, we can write

$$R_0^{\pm}(x) \equiv 1, \qquad R_\infty^{\pm}(x) = 0. \tag{2.106}$$

The initial condition on the "time" x is originated from the homogeneous initial condition (2.84) for the reflection coefficient $\Gamma_{\pm}(x)$ and from the previous requirement $R_0^{\pm}(x) \equiv 1$,

$$R_n^{\pm}(\pm L/2) = \delta_{n0}. \tag{2.107}$$

Here $\delta_{nn'}$ denotes the Kronecker delta-symbol.

2.4.2 Reflectance distribution function

Following V. L. Berezinski [17] we seek the solution of the equation (2.105) in the form

$$R_n^{\pm}(x) = \int_0^1 d\rho \tilde{P}_L^{\pm}(\rho, x)\rho^n, \qquad n = 0, 1, 2, 3 \ldots . \tag{2.108}$$

Here the variable ρ represents the local reflectance $|\Gamma_{\pm}(x)|^2$ and therefore the integration is performed over the interval $0 \le |\Gamma_{\pm}(x)|^2 \le 1$ of its possible values in line with the flow conservation law (2.78). *The function $\tilde{P}_L^{\pm}(\rho, x)$* is nothing else but *the distribution function (or probability density) of the local reflectance $|\Gamma_{\pm}(x)|^2$*.

The first initial condition on n from Eq. (2.106) yields the *normalization of the distribution function $\tilde{P}_L^{\pm}(\rho, x)$*:

$$\int_0^1 d\rho \tilde{P}_L^{\pm}(\rho, x) = 1. \tag{2.109}$$

Obviously, this formula ensures integrability of the probability density $\tilde{P}_L^{\pm}(\rho, x)$ over the variable ρ over the interval $(0, 1)$. In particular, the function $\tilde{P}_L^{\pm}(\rho, x)$ *must be integrable in the vicinity of the points* $\rho = 0$ and $\rho = 1$. This means that

$$\lim_{\rho \to 0} \rho \tilde{P}_L^{\pm}(\rho, x) = 0; \qquad \lim_{\rho \to 1} (1 - \rho) \tilde{P}_L^{\pm}(\rho, x) = 0. \tag{2.110}$$

The requirements (2.110) are direct consequences of the normalization condition (2.109). They will be used in the derivation of the differential equation for the distribution function $\tilde{P}_L^{\pm}(\rho, x)$.

So, we need to derive the equation for the function $\tilde{P}_L^{\pm}(\rho, x)$. It is noteworthy that *in what follows we consider a non-zero index n*, i.e. $n \ge 1$. In this case the normalization condition (2.109) is an additional independent condition imposed on the probability density $\tilde{P}_L^{\pm}(\rho, x)$.

Let us substitute the integral representation (2.108) into the differential-difference equation (2.105). After double integration by parts we get

$$\int_0^1 d\rho\, \rho^n \left[\pm L_{bs} \frac{\partial}{\partial x} + \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} (1 - \rho)^2 \right] \tilde{P}_L^{\pm}(\rho, x) +$$

$$+\left[n\rho^n(1-\rho)^2\tilde{P}_L^{\pm}(\rho,x)-\rho^{n+1}\frac{\partial(1-\rho)^2\tilde{P}_L^{\pm}(\rho,x)}{\partial\rho}\right]\Bigg|_0^1=0. \tag{2.111}$$

Owing to the integrability conditions (2.110) the term outside the integral vanishes and we have

$$\int_0^1 d\rho\,\rho^n\left[\pm L_{bs}\frac{\partial}{\partial x}+\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}(1-\rho)^2\right]\tilde{P}_L^{\pm}(\rho,x)=0. \tag{2.112}$$

Since the set of power functions $\{\rho^n\}$ is complete in the interval $(0,1)$, the equality (2.112) is satisfied only if the integrand is equal to zero. In this way we come to *the differential equation for the distribution function of the local reflectance*,

$$\pm L_{bs}\frac{\partial\tilde{P}_L^{\pm}(\rho,x)}{\partial x}=-\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}(1-\rho)^2\tilde{P}_L^{\pm}(\rho,x). \tag{2.113}$$

This equation is complemented by *the initial condition on the coordinate* x which results from the homogeneous ($n\neq 0$) initial condition (2.107) for the reflectance moment $R_n^{\pm}(x)$ and from the representation (2.108),

$$\tilde{P}_L^{\pm}(\rho,\pm L/2)=\delta(\rho-[+0]). \tag{2.114}$$

Recall that the symbol "+0" stands for an arbitrarily small positive quantity. *The condition on the variable* ρ *is the normalization condition* (2.109).

As it was expected, the obtained equation (2.113) for the probability density of the local reflectance belongs to a class of diffusion equations. A typical way to treat such equations in kinetic theories is to transform them into the well-known and thoroughly studied Fokker-Plank equation. To perform such transformation it is convenient to switch from the random quantity ρ, which has the meaning of the local reflectance and takes real values $0\leq\rho\leq 1$, to *a new random quantity* u by the formula

$$\rho=\frac{u-1}{u+1},\qquad\qquad 1\leq u\leq\infty. \tag{2.115}$$

By definition, the distribution function $P_L^{\pm}(u,x)$ of the random quantity u is

$$P_L^{\pm}(u,x)\equiv\tilde{P}_L^{\pm}(\rho(u),x)\frac{d\rho(u)}{du}=\frac{2}{(u+1)^2}\tilde{P}_L^{\pm}\left(\frac{u-1}{u+1},x\right). \tag{2.116}$$

This function is also normalized to one owing to the normalization condition (2.109),

$$\int_1^{\infty}duP_L^{\pm}(u,x)=1. \tag{2.117}$$

The equation (2.113) and the definitions (2.115) and (2.116) lead a new distribution function $P_L^{\pm}(u,x)$ to satisfy *the Fokker-Plank equation*,

$$\pm L_{bs}\frac{\partial P_L^{\pm}(u,x)}{\partial x}=-\frac{\partial}{\partial u}(u^2-1)\frac{\partial P_L^{\pm}(u,x)}{\partial u}. \tag{2.118}$$

The initial condition on the coordinate x *for the new probability density* $P_L^{\pm}(u,x)$ follows from the initial condition (2.114) and the definition (2.116),

$$P_L^{\pm}(u,\pm L/2)=\delta(u-[1+0]). \tag{2.119}$$

The condition on the variable ρ is the normalization condition (2.117).

The equation (2.118) with the conditions (2.119) and (2.117) is solved by the following standard method. We point out that the differential operator on the r.h.s. of the equation (2.118) is the differential operator for *the Legendre function of first kind*, $P_\nu(u)$. This means that the Legendre function $P_\nu(u)$ is the eigenfunction of this differential operator. The Legendre function at $\nu = -\frac{1}{2} + i\lambda$ is called *the cone function*, $P_{-\frac{1}{2}+i\lambda}(u)$. The cone function satisfies the equation

$$-\frac{\partial}{\partial u}(u^2-1)\frac{\partial}{\partial u}P_{-\frac{1}{2}+i\lambda}(u) = \left(\frac{1}{4}+\lambda^2\right)P_{-\frac{1}{2}+i\lambda}(u), \tag{2.120}$$

where λ is a real parameter. This function possesses some peculiarities. For instance, the equation (2.120) reveals that the cone function is an even function of the parameter λ,

$$P_{-\frac{1}{2}-i\lambda}(u) = P_{-\frac{1}{2}+i\lambda}(u). \tag{2.121}$$

Next, $P_{-\frac{1}{2}+i\lambda}(u)$ is real for $u \geq 1$. Then we note that

$$P_\nu(1) = 1. \tag{2.122}$$

Let us write down the integral representation for the cone function which we will need in further calculations,

$$P_{-\frac{1}{2}+i\lambda}(\cosh\alpha) = \frac{\sqrt{2}}{\pi}\coth(\pi\lambda)\int_\alpha^\infty dv \frac{\sin(\lambda v)}{(\cosh v - \cosh\alpha)^{1/2}}, \qquad \alpha \geq 0. \tag{2.123}$$

It is significant that for a positive parameter $\lambda \geq 0$ the functions

$$[\lambda\tanh(\pi\lambda)]^{1/2}\, P_{-\frac{1}{2}+i\lambda}(u) \tag{2.124}$$

form *a complete and orthonormal basis* in the space of functions defined in the interval $1 \leq u \leq \infty$. In other words,

$$\begin{aligned} \int_0^\infty d\lambda\,\lambda\tanh(\pi\lambda)P_{-\frac{1}{2}+i\lambda}(u)P_{-\frac{1}{2}+i\lambda}(u') &= \delta(u-u'), \\ \lambda\tanh(\pi\lambda)\int_1^\infty du P_{-\frac{1}{2}+i\lambda}(u)P_{-\frac{1}{2}+i\lambda'}(u) &= \delta(\lambda-\lambda'). \end{aligned} \tag{2.125}$$

Therefore, we can seek the solution $P_L^\pm(u,x)$ of the Fokker-Plank equation (2.118) as an expansion in the complete set (2.124),

$$\begin{aligned} P_L^\pm(u,x) &= \int_0^\infty d\lambda\,\lambda\tanh(\pi\lambda)P_{-\frac{1}{2}+i\lambda}(u)Q_L^\pm(\lambda,x), \\ Q_L^\pm(\lambda,x) &= \int_1^\infty du P_{-\frac{1}{2}+i\lambda}(u)P_L^\pm(u,x). \end{aligned} \tag{2.126}$$

Such expansion is called *the Mehler transformation* (Mehler, 1881).

Let us substitute the integral Mehler representation (2.126) for the distribution function $P_L^{\pm}(u,x)$ into the Fokker-Plank equation (2.118). After that we get the first order differential equation over the coordinate x for the Mehler transform $Q_L^{\pm}(\lambda,x)$. This equation is simply solvable. As a result we arrive at the next expression for the probability density $P_L^{\pm}(u,x)$ of the random quantity u:

$$P_L^{\pm}(u,x)=\int_0^{\infty}d\lambda\,\lambda\tanh(\pi\lambda)P_{-\frac{1}{2}+i\lambda}(u)\exp\left[-\left(\frac{1}{4}+\lambda^2\right)\frac{L\mp 2x}{2L_{bs}}\right]. \tag{2.127}$$

We now want to verify that the solution (2.127) actually satisfies the initial condition (2.119). To this end it is sufficient to present the delta-function $\delta(u-1)$ as the Mehler expansion (2.125) taking $u'=1$ and using the property (2.122) of the cone function.

Let us now use the integral representation (2.123) for the cone function entering the expression (2.127). Then we can take explicitly the integral over λ. In this way we obtain *the conventional form for the distribution function* $P_L^{\pm}(u,x)$,

$$\begin{aligned}&P_L^{\pm}(\cosh\alpha,x)=\frac{1}{\sqrt{8\pi}}\left(\frac{L\mp 2x}{2L_{bs}}\right)^{-3/2}\exp\left(-\frac{L\mp 2x}{8L_{bs}}\right)\times\\&\times\int_{\alpha}^{\infty}\frac{vdv}{(\cosh v-\cosh\alpha)^{1/2}}\exp\left[-\frac{v^2}{4}\left(\frac{L\mp 2x}{2L_{bs}}\right)^{-1}\right].\\&u=\cosh\alpha,\qquad\qquad\alpha\geq 0.\end{aligned} \tag{2.128}$$

At the end of our calculations we need to make sure that the probability density (2.128) actually satisfies the normalization condition (2.117). So, we should take the integral

$$\begin{aligned}&\int_1^{\infty}duP_L^{\pm}(u,x)=\frac{a^{3/2}}{\sqrt{8\pi}}\exp\left(-\frac{1}{4a}\right)\int_0^{\infty}d\alpha\,\sinh\alpha\times\\&\times\int_{\alpha}^{\infty}\frac{vdv}{(\cosh v-\cosh\alpha)^{1/2}}\exp\left(-\frac{av^2}{4}\right).\end{aligned} \tag{2.129}$$

Let us change the order of integration over the variables α and v. After that the integral over α is easily taken. Next, we have to change the integration variable v for $z=v/2$:

$$\begin{aligned}&\int_1^{\infty}duP_L^{\pm}(u,x)=\frac{4}{\sqrt{\pi}}a^{3/2}\exp\left(-\frac{1}{4a}\right)\times\\&\times\int_0^{\infty}dz\,z\sinh z\exp\left(-az^2\right)=1,\qquad\qquad a=2L_{bs}/(L\mp 2x).\end{aligned} \tag{2.130}$$

The integral over z is equal to the inverse value of the factor outside the integral. So, we conclude that the distribution function $P_L^{\pm}(u,x)$ in the form (2.128) is in fact normalized to one.

Thus, *we have completely solved the electron transport problem in a 1D disordered conductor by deriving the formula* (2.128) *for the distribution function* $P_L^{\pm}(u,x)$. As a matter of fact, knowing $P_L^{\pm}(u,x)$ allows us to calculate any local moment of the reflectance

and, consequently, transmittance and conductance. Setting the coordinate x equal to $\mp L/2$ ($x = \mp L/2$), we get the corresponding moments for a whole wire of length L.

For example, in accordance with the definitions (2.108), (2.115) and (2.116), *the n-th moment* (2.95) *of the local reflectance* $|\Gamma_{\pm}(x)|^2$ is expressed via the distribution function $P_L^{\pm}(u,x)$ as

$$R_n^{\pm}(x) = \int_1^{\infty} du P_L^{\pm}(u,x) \left(\frac{u-1}{u+1}\right)^n, \qquad n = 0,1,2,3\ldots\ . \tag{2.131}$$

2.5 Moments of Transmittance

Let us introduce *the n-th moment*, $T_n^{\pm}(x)$, *of the local transmittance* $|\Upsilon_{\pm}(x)|^2$ in line with the general definition for the moments of a random quantity and according to the flow conservation law (2.78):

$$T_n^{\pm}(x) \equiv \langle |\Upsilon_{\pm}(x)|^{2n} \rangle = \langle \Big[1 - |\Gamma_{\pm}(x)|^2\Big]^n \rangle, \qquad n = 0, \pm 1, \pm 2, \pm 3\ldots\ . \tag{2.132}$$

We can find this quantity with the aid of the probability density $P_L^{\pm}(u,x)$ (2.128).

It is interesting to emphasize that in contrast to the n-th reflectance moment $R_n^{\pm}(x)$, the transmittance moment $T_n^{\pm}(x)$ may take not only positive but also negative values of the number n. Indeed, the integral in the representation (2.108) for $R_n^{\pm}(x)$ converges at the lower limit $\rho = 0$ only for $n \geq 0$. At the same time, by expanding Eq. (2.132) in power series, the moments $T_n^{\pm}(x)$ even with negative numbers n can be expressed via the moments $R_m^{\pm}(x)$ of positive numbers m only. This means that in order to calculate the transmittance moment $T_n^{\pm}(x)$ with a number n of either sign we can use the same distribution function $P_L^{\pm}(u,x)$.

The quantity $T_n^{\pm}(x)$ is determinative characteristic of the system under consideration. Indeed, the transmittance moment of zeroth order, $T_0^{\pm}(x)$, must be equal to one by the definition (2.132) and the normalization condition (2.117),

$$T_0^{\pm}(x) = 1. \tag{2.133}$$

According to the expression (2.94) and the definition (2.132), the first moment $T_1^{\pm}(\mp L/2)$ taken at the wire ends $x = \mp L/2$ coincides with *the average dimensionless conductance*,

$$\langle T(L) \rangle = T_1^{\pm}(\mp L/2). \tag{2.134}$$

Then, the minus-first moment $T_{-1}^{\pm}(\mp L/2)$ gives us *the average dimensionless resistance* $\langle T^{-1}(L) \rangle$ of a 1D disordered conductor,

$$\langle T^{-1}(L) \rangle = T_{-1}^{\pm}(\mp L/2). \tag{2.135}$$

Second moments, $T_2^{\pm}(\mp L/2)$ and $T_{-2}^{\pm}(\mp L/2)$, specify dispersions of the conductance and resistance, respectively.

So, according to definitions (2.132) for the local transmittance moment and (2.108) for the reflectance distribution function, the local transmittance $|\Upsilon_{\pm}(x)|^2$ is presented by the following random functions:

$$1-\rho=\frac{2}{u+1}=\cosh^{-2}(\alpha/2). \tag{2.136}$$

This leads to the following formula for the n-th moment $T_n^{\pm}(x)$,

$$T_n^{\pm}(x)=\int_1^{\infty}duP_L^{\pm}(u,x)\frac{2^n}{(u+1)^n}=\int_0^{\infty}\frac{d\alpha\,\sinh\alpha}{\cosh^{2n}(\alpha/2)}P_L^{\pm}(\cosh\alpha,x). \tag{2.137}$$

Let us substitute here the expression (2.128) for the distribution function $P_L^{\pm}(\cosh\alpha,x)$. It is convenient to introduce the notation a from Eq. (2.130) to manage intermediate transformations. After the substitution we change the integration variables α and v for $t=\alpha/2$ and $z=v/2$:

$$T_n^{\pm}(x)=\frac{4}{\sqrt{\pi}}a^{3/2}\exp\left(-\frac{1}{4a}\right)\int_0^{\infty}\frac{dt\,\sinh t}{\cosh^{2n-1}t}\int_t^{\infty}\frac{zdz\,\exp(-az^2)}{(\cosh^2 z-\cosh^2 t)^{1/2}}. \tag{2.138}$$

Then we change the order of integration over the variables t and z:

$$T_n^{\pm}(x)=\frac{4}{\sqrt{\pi}}a^{3/2}\exp\left(-\frac{1}{4a}\right)\int_0^{\infty}zdz\,\exp(-az^2)\int_0^{z}\frac{dt\,\sinh t\cosh^{1-2n}t}{(\cosh^2 z-\cosh^2 t)^{1/2}}. \tag{2.139}$$

Finally we change the integration variable t by y with the formula $\cosh t=\cosh z/\cosh y$. In this way we get *the ultimate expression for the n-th moment of the local transmittance*,

$$\begin{aligned}T_n^{\pm}(x) &= \frac{4}{\sqrt{\pi}}\left(\frac{L\mp 2x}{2L_{bs}}\right)^{-3/2}\exp\left(-\frac{L\mp 2x}{8L_{bs}}\right)\times\\ &\times \int_0^{\infty}\frac{zdz}{\cosh^{2n-1}z}\exp\left[-z^2\left(\frac{L\mp 2x}{2L_{bs}}\right)^{-1}\right]\int_0^{z}dy\,\cosh^{2(n-1)}y.\end{aligned} \tag{2.140}$$

The formula (2.140) is the most suitable one for analysis.

2.5.1 Average dimensionless conductance

Let us calculate *the average of the dimensionless conductance* (2.2). According to the relation (2.134), we take the number n equal to one and set the coordinate $x=\mp L/2$ in the expression (2.140) for the local transmittance moment $T_n^{\pm}(x)$. The integral over y is calculated elementary and we get a relatively simple and conventional result

$$\langle T(L)\rangle=\frac{4}{\sqrt{\pi}}\left(\frac{L}{L_{bs}}\right)^{-3/2}\exp\left(-\frac{L}{4L_{bs}}\right)\int_0^{\infty}\frac{z^2dz}{\cosh z}\exp\left[-z^2\left(\frac{L}{L_{bs}}\right)^{-1}\right]. \tag{2.141}$$

The formula (2.141) specifies completely the averaged transport characteristic of a 1D disordered conductor. Depending on the ratio between the wire length L and the backscattering mean free path L_{bs}, it could have the following *asymptotics*:

$$\langle T(L)\rangle \approx 1 - L/L_{bs} \qquad \text{if} \qquad L/L_{bs} \ll 1;$$
$$\langle T(L)\rangle \approx \frac{\pi^{5/2}}{2}\left(\frac{L}{L_{bs}}\right)^{-3/2}\exp\left(-\frac{L}{4L_{bs}}\right) \qquad \text{if} \qquad L_{bs}/L \ll 1. \tag{2.142}$$

2.5.2 Average dimensionless resistance

Let us consider the average dimensionless resistance $\langle T^{-1}(L)\rangle$ of a 1D disordered conductor. According to the definition (2.135) we set $n = -1$ and $x = \mp L/2$ in the equation (2.140). After that the integrals over y and z are taken exactly. So, *the average dimensionless resistance* is represented by the surprisingly simple formula,

$$\langle T^{-1}(L)\rangle = \frac{1}{2}\left[1 + \exp\left(\frac{2L}{L_{bs}}\right)\right]. \tag{2.143}$$

The minus-second moment, $T^{\pm}_{-2}(\mp L/2)$, which specifies the dispersion of the resistance has a simple form too,

$$\langle T^{-2}(L)\rangle \equiv T^{\pm}_{-2}(\mp L/2) = \frac{1}{6}\left[2 + 3\exp\left(\frac{2L}{L_{bs}}\right) + \exp\left(\frac{6L}{L_{bs}}\right)\right]. \tag{2.144}$$

2.6 Brief Discussion

The results obtained, in particular Eqs. (2.141) – (2.144), *agree absolutely with concepts of the strong localization theory for 1D disordered conductors.*

First of all, it is necessary to emphasize once more that, unlike the spectral single-particle properties, *the transport properties of a 1D disordered wire* do not depend on the total (outgoing) mean free path L_{ts} of electrons. They *are completely specified by the backscattering length L_{bs} only* while the forward electron scattering length L_{fs} is canceled. Indeed, this is only L_{bs} that enters all the transport characteristics from the distribution function $P^{\pm}_{L}(u, x)$ to the average conductance and resistance.

If the wire length L is much smaller than the backscattering length L_{bs}, then the regime of *ballistic transport* is realized. In this case the dimensionless conductance and resistance are equal to one in zeroth-order approximation in the parameter $L/L_{bs} \ll 1$. Electron relaxation enters only starting from the first-order approximation. This statements result from the asymptotic (2.142) for the conductance and the exact expression (2.143) for the resistance.

The localization regime arises in a sufficiently long wire when its length L exceeds $4L_{bs}$. Here the asymptotic (2.142) shows an exponential decrease of the average conductance over the spatial scale $4L_{bs}$ which is four times as larger than the backscattering length L_{bs}. This scale is called *the localization length L_{loc}*,

$$L_{loc} = 4L_{bs}. \tag{2.145}$$

The formula (2.143) describes an exponential growth of the average resistance with growing the conductor length L.

We would like to point out the following two facts. First, from comparison of the expressions for the average conductance with those for the resistance we conclude that the average resistance is not equal to the inverse average conductance,

$$\langle T^{-1}(L)\rangle \neq \langle T(L)\rangle^{-1}. \tag{2.146}$$

Second, from the equalities (2.143) and (2.144) for the average resistance and its second moment it follows that the dispersion of the resistance is of the order of the squared average resistance itself (or the r.m.s. deviation of the resistance is of the order of the average resistance itself). These two facts mean that both *the conductance and the resistance are not self-averaged quantities*. Hence, depending on the wire length L, their behavior "on average" must be accompanied by fluctuations called *mesoscopic oscillations*.

Thus, *a 1D disordered conductor represents an example of a mesoscopic system*.

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