Flowbox Tilings for Computing Closed Invariant Manifolds

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\[ u' = F(u, \lambda) \]

\( M(\lambda) \) is compact, connected, and without boundary (closed)

if \( u(0) \in M(\lambda), \ u(t) \in M(\lambda) \) for all \( t > 0 \)

Given \( M(\lambda - \Delta \lambda) \), find \( M(\lambda) \). Existence requires “hyperbolicity”
Flow onto \( M \) dominates flow on \( M \).

1-d invariant manifolds

Fixed point (0-d)

Periodic Motion
invariant circle

Heteroclinic Motion
invariant line segment w/ fixed points at ends

Homoclinic Motion
invariant circle w/ a fixed point
Higher dimensional invariant manifolds

Quasiperiodic motion

invariant torus

Motions on manifolds which are not closed

Stable/Unstable Manifolds of fixed points, periodic motions, ...
One dimensional invariant manifolds

Can use time to parameterize.

Remove the shift using a nearby "reference" manifold (v on M(λ-Δλ)).

TPBVP

\[
\int_0^1 \left( v'(t) \cdot (u(t) - v(t)) \right) \, dt = 0
\]

\[
u'(t) = T \, F(u(t), \lambda)
\]

\[
\begin{align*}
\text{u(1)=u(0)} & \\
\text{u(0) on unstable mf.} & \\
\text{u(1) on stable mf.}
\end{align*}
\]
Any surface (curve) $\sigma(s)$ transverse to the flow generates an invariant manifold.

$$
M= \{ \ u \mid u(t,s) = \sigma(s) + \int_{t'\neq 0}^t F(u(t'))dt', \text{for some } t \in [-\infty, \infty] \text{ and } s \}$$
Stable/Unstable Manifolds

\[ M = \{ u \mid u(t,s) = \sigma(s) + \int_{t'=0}^{t} F(u(t'))dt', \text{ for some } t \in [-\infty, \infty] \text{ and } s \} \]

For other manifolds must find \( \sigma(s) \).
If every point on $M$ lies on a trajectory beginning and ending on the transversal it is a global transversal.

$$u(t,s) = \sigma(s) + \int_{t'=0}^{t} F(u(t')) dt'$$

$M = \sigma \times S^1$  
G. Birkhoff 1920's
Can use the integral of $F$ to construct a return map.

(TPBVP's to integrate $F$?)

$\mathcal{G}(s)$ $\mathcal{G}(\tau(s))$

Circle Map
Find $M$ by requiring that $F(u)$ lie in the tangent plane, with $\sigma$ as boundary conditions.

\[
\alpha f_{\theta_0} + \beta f_{\theta_1} = F(f(\theta_0, \theta_1, \lambda), \lambda) + \text{periodic bc}
\]

Tangency constraints:

\[
f_{\theta_0} \cdot (f(\theta_0, \theta_1, \lambda) - f(\theta_0, \theta_1, \lambda - \Delta\lambda)) = 0
\]

Phase constraints:

\[
f_{\theta_1} \cdot (f(\theta_0, \theta_1, \lambda) - f(\theta_0, \theta_1, \lambda - \Delta\lambda)) = 0
\]
Global transversals

If every point on M lies on a trajectory beginning and ending on the transversal it is a global transversal.

For certain topologies, closed, connected global transversals cannot exist for any flow. (e.g. sphere – must have at least one fixed point).

For topologies where they can exist (e.g. torus, surface of Klein bottle) there are flows which do not have one.

Bender & Orszag

Reeb flow (?)
Other kinds of global transversals.
Global transversals from Flow Box Tilings

Flow box/Stream Tube

Inset

Outset

Inset
Global transversals from Flow Box Tilings

(not closed, not connected)
Piecewise smooth return map:
Figure 1. The tower for the Fibonacci map.

Bruin, “Combinatorics of the Kneading Map”
IJBC 1995, 5(5)

Figure 3. The tower $N$ for $T^3$ with an irrational flow. In the left hand picture, the subset of $\Sigma$ where $\tau_+$, and hence the first return map $h$, is discontinuous is shown as the dotted lines. On each region on which $h$ is continuous is labeled. In the right hand picture, the identification resulting from the first return map $h$ is labeled. The identification on the rest of $N$ is forced by the flow.

Basener, “Global cross sections and minimal flows”
Topology and its Applications, 2002, 121(3)
Constructing a flowbox covering

**Fixed point free regions**: Easy to show can be covered.
Choose a point in an uncovered part.
Use a disc orthogonal to the flow as the inset.
Repeat.

**Near fixed points** the outset will become very small.
Place a ball around the fixed point

Can construct a flow box tiling from this
Can construct a flow box tiling from this

Straighten flow boxes about the center of the fat trajectory.

Not going to actually do this
The global transversal is the collection of insets and outsets (curves)

Inset

Outset

Not going to actually do this
Overlapping spheres have a plane containing their intersection.

The points on the "center-side" make a Voronoi cell.

The dual is an edge between center if plane exists. A list of "nearest" neighbors. (and a triangulation)
Fat trajectory

$M(\lambda - \Delta \lambda)$ is known, cover it with trajectory fragments.

Theoretically could find a flowbox covering and then a tiling.

Extend the sections off of $M(\lambda - \Delta \lambda)$ to define a return map, find $M(\lambda)$.

Example from Aronson, Doedel and Othmer, 1987

Instead will use these trajs. to write a system of ODEs for $M(\lambda + \Delta \lambda)$
For each flow box a two point boundary value problem

\( u(1) \) must lie on \( M(\lambda) \)

\( u(0) \) must lie on \( M(\lambda) \) above \( v(0) \)

\( u(0) \) must lie on \( M(\lambda) \) above \( v(0) \)

Don't know \( M \) yet.
Use nearest neighbors, and interpolate
Need smooth interpolation on the interior of a convex polyhedra.
A (large) set of coupled TPBVPs

\[ u_i'(s) = \tau_i \ F(u_i(s), \lambda + \Delta \lambda) \quad s \in [0,1] \]

\[ \Phi^T_0 \ ( u_i(0) - v_i(0) ) = 0 \quad \text{Initial point lies above } v(0) \]

\[ \Phi^T_1 \ ( u_i(1) - v_i(1) ) . F = 0 \quad \text{End point lies beside } v(1) \]

\[ (I - \Phi_1 \Phi^T_1)(u_i(1) - v_i(1)) = a_i(v_i(1), s_i(1)) \quad \text{Endpoint lies on interpolant} \]

\[ \int_0^1 F^T(v_i(s)) \ ( u_i(s) - v_i(s) ) \ ds = 0 \quad \text{Phase condition} \]

\[ s_i(s) = \Phi^T(u_i(s)) \ ( v_i(s) - u_i(s) ) \quad \text{unknown.} \]

\[ a(u, s) \text{ interpolates near } u, \text{ at } s \text{ in TM.} \]
182 Trajectories, 22,900 points
longest trajectory has 223 points, shortest has 4.

Example from Aronson, Doedel and Othmer, 1987
2 coupled oscillators w/ cubic coupling
218 trajectories, 4,400 points
longest 138, shortest 2.

2 fixed points
both sources
1 stable periodic orbit

218 trajectories, 4,400 points
longest 138, shortest 2.
4 fixed points
2 saddles
2 sources
1 stable periodic orbit

758 trajectories, 63,900 points
Challenges

Implement solver for this system.

Improve the covering, fewer short trajectories, more robust.

Remeshing.
Summary

A global transversal allows a return map to be written for $M(\lambda)$

For most flows on most manifolds there is no closed global transversal

Use a different kind of global transversal, made up of the insets and outsets of a flowbox tiling of $M$.

Discretize by representing $M$ with one trajectory per flowbox tile. Write TPBVP.

Acts as a mesh that conforms to the flow on $M$.

Large system of ODE's

For one dimension

\[ v(t) \]

\[ \tau \quad \lambda \]

Boundary Conditions

Integral Constraints

Arclength Constraints
For set of trajs.
Integral Constraints
Arclength Constraints
Boundary Conditions

$\int$ "$v(t)$" $\tau$ $\lambda$

ODE