The Schröder Functional Equation and the Numerical Approximation of the Invariant Densities of Chaotic Maps


José-Rubén Luévano

Universidad Autónoma Metropolitana, Unidad Azcapotzalco, México, D.F., MÉXICO

May 17, 2013
The logistic map $T(X) = 4x(1 - x)$

Discrete chaos

$x \in [0, 1]$

\[ X_{n+1} = 4X_n(1 - X_n) = T(X_n), \quad (1) \]

as it is well known

\[ T(X) = \sin^2(2 \arcsin \sqrt{X}), \quad (2) \]

i.e., $T$ is conjugated to a circle map $\theta_{n+1} = 2\theta_n$, mod $\pi$.

Also, the iterates of $T$ are given by

\[ T^n(X) = \sin^2(2^n \arcsin \sqrt{X}), \quad (3) \]
Ernest Schröder (1841-1902)

Figure: ‘Ueber iterirte Functionen. Math. Ann. 1869’
Schröder’s equation
Global Newton’s method (to find the zeros of a given function)

\[ T : I \to I, \text{ where } I \subseteq \mathbb{R} \text{ and } \lambda \in \mathbb{R}, \]

\[ \Phi(T(X)) = \lambda \Phi(X) \quad (4) \]

then (at least locally)

\[ T(X) = \Phi^{-1}(\lambda \Phi(X)), \quad (5) \]

(i.e., for a given neighborhood of \( X \).)
Some exactly solvable systems

\[ T(X) = \Phi^{-1}(\lambda \Phi(X)), \quad (6) \]

| \( T(x) \)                           | \( \Phi^{-1} \)     | \( \rho(x) \)                     | \( |\lambda| \) |
|--------------------------------------|---------------------|-----------------------------------|-----------------|
| \( 4x(1 - x) \)                      | \( \sin^2(\pi x) \) | \( \frac{1}{\pi \sqrt{x(1-x)}} \) | \( 2 \)         |
| \( \cos(r \cos^{-1}(x)) \)          | \( \cos(x) \)       | \( \frac{1}{\pi \sqrt{1-x^2}} \)  | \( r \)         |
| \( \frac{1}{2}(x - \frac{1}{x}) \)  | \( \cot(x) \)       | \( \frac{1}{\pi (1+x^2)} \)       | \( 2 \)         |
Let us consider the Schröder equation:

$$\Phi(T(X)) = \lambda \Phi(X),$$  \hspace{1cm} (7)

Now, taking the derivative of this equation, we have

$$T'(X)\Phi'(T(X)) = \lambda \Phi'(X)$$  \hspace{1cm} (8)

and, after taking the absolute value of this one

$$\left| T'(X) \right| \left| \Phi'(T(X)) \right| = \left| \lambda \right| \left| \Phi'(X) \right|,$$  \hspace{1cm} (9)

we obtain the invariant density of $T$, which is denoted by $\rho(X)$:

$$\rho(X) = \left| \Phi'(X) \right|.$$  \hspace{1cm} (10)
Now, $\rho(X) = |\Phi'(X)|$ then

$$|T'(X)|\rho(T(X)) = |\lambda|\rho(X),$$

(11)

Now, we consider the following iteration process

$$\rho(T(X_n)) = \frac{|\lambda|}{|T'(X_n)|}\rho(X_n),$$

(12)

$$X_{n+1} = T(X_n),$$

(13)
Numerical method

Laplace’s method (George Boole (1815-1864), exposed in his book: “Calculus of finite differences” (1860))

Denoting by \( \rho(X_n) = \rho_n \) and \( \rho(T(X_n)) = \rho_{n+1} \) then we have the final form for the last iteration process as

\[
\rho_{n+1} = \frac{|\lambda|}{|T'(X_n)|} \rho_n \tag{14}
\]

\[
X_{n+1} = T(X_n) \tag{15}
\]
Difference-equations are a particular species of functional equations, ... . And the most general method of solving functional equations ... consists in reducing them to difference-equations. Laplace has given such a method, ...

George Boole (1815-1864), exposed in his book: ”Calculus of finite differences” (1860)) (Project Gutenberg)
Invariant density

Logistic map \( X_{n+1} = 4X_n(1 - X_n), \ X_n \in [0, 1] \)

Figure: the invariant density \( \rho(x) = \frac{1}{\pi \sqrt{x(1 - x)}} \)
Lyapunov exponent, $\Lambda$

Numerical method

from the ergodic theory:

$$\Lambda = \frac{1}{n} \sum_{k=0}^{n-1} \ln |T'(X_k)|$$

(16)

and, we can see from the Schröder equation that the invariance of $\rho$ under $T$ imply that

$$\Lambda = \ln |\lambda|.$$ 

(17)
Invariant density

Cubic Chebyshev map \( X_{n+1} = X_n(3 - 4X_n^2), \ X_n \in [-1, 1] \)

**Figure:** the invariant density \( \rho(x) = \frac{1}{\pi \sqrt{1 - x^2}} \)
Invariant density
Cubic Chebyshev map $X_{n+1} = X_n(3 - 4X_n^2), \ X_n \in [-1, 1]$
Lyapunov exponent
Cubic Chebyshev map $X_{n+1} = X_n(3 - 4X_n^2)$, $X_n \in [-1, 1]$
Invariant density

Gauss map $X_{n+1} = e^{-8X_n^2} - 0.27$, $X_n \in \mathbb{R}$

Figure: the invariant density $\rho(x) = ?$
Invariant density

Gauss map: $X_{n+1} = e^{-8X_n^2} - 0.27$, $X_n \in \mathbb{R}$

Figure: the invariant density $\rho(x) =$?
Lyapunov exponent

Gauss map $X_{n+1} = e^{-8X_n^2} - 0.27, X_n \in \mathbb{R}$

Figure: the invariant density $\rho(x) =$ ?
Circle map near intermittency

\[ X_{n+1} = X_n + \Omega - \frac{K}{2\pi} \sin(2\pi X_n), \mod 1 \]

![Graph showing the circle map near intermittency with parameters \( \Omega = 0.91 \) and \( K = 0.545 \).]
Invariant density near intermittency

Circle map $X_{n+1} = X_n + \Omega - \frac{K}{2\pi} \sin(2\pi X_n), \text{ mod } 1$

**Figure**: $\approx$ invariant density $\rho(x) = \rho$: by histogram with $N = 9000$ points, points in blue colour. By numerical iteration, points in red.
Lyapunov exponent near intermittency

Circle map $X_{n+1} = X_n + \Omega - \frac{K}{2\pi} \sin(2\pi X_n)$, mod 1
Schroder’s equation as an *adjoint* problem to Frobenius-Perron equation

the Frobenius-Perron operator:

\[ \rho(X) = \int \delta(X - T(Y))\rho(Y)dY, \quad (18) \]

\[ = \sum_{j=0}^{r} \frac{\rho(T_j^{-1}(X))}{|T'(T_j^{-1}(X))|}, \quad (19) \]

and the Koopman operator

\[ U[\Phi(Y)] = \int \delta(X - T(Y))\Phi(X)dX, \quad (20) \]

\[ \Phi(T(Y)) = \lambda \Phi(Y), \quad (21) \]
"Amo el canto del Zenzontle, pájaro de 400 voces, 
Amo el color del jade, 
y el fragante perfume de las flores, 
Pero amo más a mi hermano, el Hombre”
Nezahualcóyotl (1402-1472)
el más importante Poeta Azteca

"I love the sing of the Zenzontle, bird having 400 voices, 
I love the colour of jade, 
and the fragrant perfume of flowers, 
But, the most I love is my brother, the Man”
Nezahualcóyotl (1402-1472)
the most important Poet of the Azteca civilization
The Schröder Functional Equation and the Numerical Approximation of the Invariant Densities of Chaotic Maps

References


- George Boole ... reprinted by DOVER