Física Estadística I
Tarea 04: Gases Ideales Cuánticos – Gas de Bose-Einstein

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Nombre del Estudiante: ____

Problema 1 *Entropy of an ideal gas*

Show that the entropy of an ideal gas in thermal equilibrium is given:

1. In the case of bosons by the following expression,

$$S = k_B \sum_k \left[\langle \hat{n}_k + 1 \rangle \ln \langle \hat{n}_k + 1 \rangle - \langle \hat{n}_k \rangle \ln \langle \hat{n}_k \rangle \right].$$

2. In the case of fermions as:

$$S = k_B \sum_k \left[- \langle 1 - \hat{n}_k \rangle \ln \langle 1 - \hat{n}_k \rangle - \langle \hat{n}_k \rangle \ln \langle \hat{n}_k \rangle \right].$$

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Problema 2 *Occupation number's fluctuation*

1. Derive, for all three statistics, the following expressions for the occupation number's fluctuation, $\sigma^2_{\hat{n}_k} = \langle \hat{n}_k^2 \rangle - \langle \hat{n}_k \rangle^2$:

$$\begin{aligned} \text{Bose-Einstein:} \quad & \sigma^2_{\hat{n}_k}\Big|_{BE} = \langle \hat{n}_k \rangle + \langle \hat{n}_k \rangle^2 . \\ \text{Fermi-Dirac:} \quad & \sigma^2_{\hat{n}_k}\Big|_{FD} = \langle \hat{n}_k \rangle - \langle \hat{n}_k \rangle^2 . \\ \text{Maxwell-Boltzmann:} \quad & \sigma^2_{\hat{n}_k}\Big|_{MB} = \langle \hat{n}_k \rangle . \end{aligned}$$

2. Show that, quite generally:

$$\sigma^2_{\hat{n}_k} = \langle \hat{n}_k^2 \rangle - \langle \hat{n}_k \rangle^2 = k_B T \left. \frac{\partial \langle \hat{n}_k \rangle}{\partial \mu} \right|_T .$$

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Problema 3 *Thermodynamic relationships for ideal Bose gas*

1. For the ideal Bose gas it was demonstrated that:

$$\frac{1}{\zeta}\left(\frac{\partial \zeta}{\partial T}\right)_V = -\frac{3}{2T}\frac{g_{3/2}(\zeta)}{g_{1/2}(\zeta)},$$

   thus, show the following:

$$\frac{1}{\zeta}\left(\frac{\partial \zeta}{\partial T}\right)_p = -\frac{5}{2T}\frac{g_{5/2}(\zeta)}{g_{3/2}(\zeta)}.$$

2. Hence, also show that,

$$\gamma = \frac{C_p}{C_V} = \frac{(\partial\zeta/\partial T)_p}{(\partial\zeta/\partial T)_V} = \frac{5}{3}\frac{g_{5/2}(\zeta)g_{1/2}(\zeta)}{[g_{3/2}(\zeta)]^2}.$$

3. Check that, as T aproaches T_c from above, both γ and C_p diverge as $(T-T_c)^{-1}$.

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Problema 4 *Bose-Einstein condensation*

Consider a three-dimensional gas of bosons for which the single-particle energy is given by,

$$\epsilon_{\mathbf{k},n} = \frac{\hbar^2\mathbf{k}^2}{2m} + \alpha n,$$

where α is a positive constant and $n = -j, \ldots, j$ is an integer number.

1. Find that the expressions for the pressure p and the mean number of particles N in terms of the temperature T and the fugacity $\zeta = e^{\beta\mu}$ are:

$$N(T,V,\mu) = \sum_{n=-j}^{j} \frac{V}{\lambda^3} g_{3/2}(\zeta e^{-\beta\alpha n}) + \frac{\zeta e^{\beta\alpha j}}{1-\zeta e^{\beta\alpha j}},$$

$$p(T,V,\mu) = \frac{k_B T}{\lambda^3}\sum_{n=-j}^{j} g_{5/2}(\zeta e^{-\beta\alpha n}).$$

2. Write down the condition for Bose-Einstein condensation, in particular, find that the conditions for ζ and N_ϵ^{MAX} are given by:

$$\zeta = e^{-\beta\alpha j}, \quad N_\epsilon^{MAX} = \sum_{l=0}^{2j} \frac{V}{\lambda^3} g_{3/2}(e^{-\beta\alpha l}).$$

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Problema 5 *Black-Body thermodynamics*

1. Find expressions for the pressure p, energy density E/V, entropy density S/V and specific heat at constant volume C_V of black-body radiation in a d-dimensional cavity at temperature T:

$$p = a_d I_d (k_B T)^{d+1}, \qquad E/V = d a_d I_d (k_B T)^{d+1},$$
$$S/V = (d+1) a_d I_d k_B (k_B T)^d, \qquad C_V = d(d+1) a_d I_d k_B V (k_B T)^d,$$

where the quantities a_d and I_d are given by:

$$a_d = \frac{d-1}{2^{d-1}} \frac{1}{\Gamma(d/2)(\sqrt{\pi} c\hbar)^d}, \qquad I_d = \frac{1}{d}\Gamma(d+1)\mathcal{Z}(d+1).$$

2. Evaluate these quantities explicitly for $d = 3$.

Hint: $\epsilon_k = c\hbar|\mathbf{k}|$, where the reciprocal vector $\mathbf{k}$ is quantized as following: $\mathbf{k} = (\pi/L)\mathbf{n}$, and $\mathbf{n}$ is a d-dimension vector of positive integers.

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Problema 6 *Phonons on a linear molecule*

A linear molecule composed of n identical atoms may be regarded as having a phonon spectrum given by:

$$\omega_r = \omega_c \text{Sen}\left(\frac{\pi}{2n} r\right) \quad \forall \quad r = 1, 2, \ldots, (n-1),$$

where ω_c is a characteristic vibrational frequency of the molecule. Show that this model leads the following behaviors of the specific heat C_V:

1. C_V varies as T at low temperatures.

2. C_V tends to the limiting value $(n-1)k_B$ at high temperatures.

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