Física Estadística I
Tarea 05: Gases Ideales Cuánticos – Gas de Fermi-Dirac

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25 Mayo 2020

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Problema 1 *Fermi gas compressibility*

Calculate the following properties of a nearly-degenerate ideal Fermi gas, up to T^2-order terms.

1. The Fermi energy of the gas, ϵ_F.

2. The chemical potential, μ.

3. The pressure, p.

4. The isothermal compressibility κ, given by,

$$\kappa = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_T .$$

Hint: the energy dispersion of an ideal gas is $\epsilon_k = \hbar^2 \mathbf{k}^2/2m$.

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Problema 2 *Constant density of states*

Let the density of states of the electrons in some sample be assumed to be a constant D for $\epsilon > 0$ ($D = 0$ for $\epsilon < 0$) and the total number of electrons be equal to N.

1. Calculate the Fermi energy ϵ_F.

2. Show that the specific heat is proportional to T when the system is nearly-degenerate ($T \ll T_F$).

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Problema 3 *Relativistic Fermi gas*

Consider a relativistic Fermi gas, with one-particles energies ϵ_p given as:

$$\epsilon_p = mc^2 \left[\sqrt{1 + \left(\frac{p}{mc} \right)^2} - 1 \right],$$

where the rest mass of the particles has been substracted. For the degenerate case, calculate the following:

1. The Fermi momentum p_f,

$$p_f = \left[\frac{3h^3}{4\pi g_s} \frac{N}{V} \right]^{1/3}.$$

2. The pressure $\mathcal{P}$,

$$\mathcal{P} = g_s \frac{4\pi}{3h^3}(mc^2) \int_0^{p_f} \frac{p^2 (p/mc)^2}{\sqrt{1+(p/mc)^2}} dp = g_s \frac{\pi}{6h^3}(m^4c^5) A(y_f),$$
$$\forall \;\; A(y_f) = \sqrt{1+y_f^2}(2y^3 - 3y) + 3\text{ArcSenh}y \;\; \& \;\; y_f = \frac{p_f}{mc}.$$

3. The energy E,

$$E = g_s \frac{4\pi V}{h^3}(mc^2) \int_0^{p_f} p^2 \left[\sqrt{1+(p/mc)^2} - 1 \right] dp = g_s \frac{\pi V}{6h^3}(m^4c^5) B(y_f),$$
$$\forall \;\; B(y_f) = 8y^3 \left[\sqrt{1+y_f^2} - 1 \right] - A(y_f).$$

4. Obtain in the nonrelativistic limit ($p \ll mc$) the following expression:

$$\mathcal{P} = \frac{2}{3}\frac{E}{V}\left[1 - \frac{5}{28}(p_f/mc)^2 \right].$$

5. Calculate for the ultrarelativistic limit ($p \gg mc$) the next relationship,

$$\mathcal{P} = \frac{1}{3}\frac{E}{V}\left[1 + \frac{4}{3}\frac{1}{(p_f/mc)} - \frac{2}{9}\frac{1}{(p_f/mc)^2} \right].$$

Hint: For the limits analysis, you could obtain the proper expansions of $A(y_f)$ and $B(y_f)$ by using tables, references, or the computer.

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Problema 4 *Two Fermi-gases*

A three-dimensional container of fixed volume V is divided into two compartments by a wall which is freely movable and heat-conducting, but impermeable to gas particles. Both compartments contain non-relativistic ideal Fermi-gases, one contains N particles of gas A, which has spin $s_A = 1/2$, and the other contains N particles of a gas B, which has spin $s_B = 3/2$. Consider that the two species have indentical masses $m_A = m_B = m$. Considering the conditions for this system to be in thermal ($T_A = T_B$) and mechanical ($p_A = p_B$) equilibrium, use them to find the volume ratio V_A/V_B at:

1. High-temperatures: $V_A/V_B = 1$.
2. Degenerate-Fermi gas ($T = 0$ K): $V_A/V_B = 2^{2/5}$.
3. Low-temperatures:

$$\frac{V_A}{V_B} = r\left\{1 - \frac{\pi^2}{4}\left(\frac{4}{3\sqrt{\pi}}\frac{V}{N}\right)^{4/3}\left(\frac{2m\pi}{h^2}\right)^2 (k_B T)^2\left[\left(\frac{g_B}{r+1}\right)^{4/3} - \left(\frac{r g_A}{r+1}\right)^{4/3}\right]\right\},$$

   where $r = 2^{2/5}$, $g_B = 4$, and $g_A = 2$.

Hint: the chemical potentials are different ($\mu_A \neq \mu_B$).

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